The Maxwell – Exponential Distribution: Theory and Application to Lifetime Data

Usman Aliyu Abdullahi[1], Ahmad Abubakar Suleiman[1], Aliyu Ismail Ishaq[2], Abubakar Usman[2] and Aminu Suleiman[1]

[1]*Department of Statistics, Faculty of Computing and Mathematical Science, Kano University of Science and Technology Wudil, Nigeria*

[2]*Department of Statistics, Faculty of Physical Science, Ahmadu Bello University Zaria, Nigeria*

**Corresponding author: usiyamma@gmail.com*

Abstract

Two parameters Maxwell – Exponential distribution was proposed using the Maxwell generalized family of distribution. The probability density function, cumulative distribution function, survival function, hazard function, quantile function, and statistical properties of the proposed distribution are discussed. The parameters of the proposed distribution have been estimated using the maximum likelihood estimation method. The potentiality of the estimators was shown using a simulation study. The overall assessment of the performance of Maxwell - Exponential distribution was determined by using two real-life datasets. Our findings reveal that the Maxwell – Exponential distribution is more flexible compared to other competing distributions as it has the least value of information criteria.

Keywords: Maxwell – Exponential, Maxwell generalized family, maximum likelihood estimator and Quantile function.

1. Introduction

The real lifetime datasets are so enormous to the extent that the existence lifetime distributions that are used in modeling lifetime datasets are not enough to handle all the present data (Suleiman et al., 2020). These prompt scientists from all fields working tirelessly to proposed new distributions that will be used in modeling present and past datasets, by improving the flexibility of the new distributions that were derived from an existing family of distributions.

There are various approaches used in proposing lifetime distributions as studied by many researchers (Alzaatreh et al., 2013; Bourguignon et al., 2014). Several researchers used these methods to propose new distribution such as (Yousof et al., 2016; Alizadeh et al., 2017; Cordeiro et al., 2017; Gomes et al., 2017; Jamal et al., 2017; Ahsan Ul Haq et al., 2018; Muhammad et al., 2018; Nadia and Lamyaa, 2018).

Weibull and Gamma distributions are the most widely two parametric distributions for analyzing most of the lifetime datasets (Weibull W. 1939; Johnson et al., 1994), it also has more application apart from the modeling of lifetime data (Klinken J. and Van A., 1961; Alexander G. N., 1962; Jackson O. A. Y., 1963). The Gamma distribution with two parameters namely scale and shape parameters is more flexible to model real positive data, it's shape parameter enable it to possess increasing and decreasing failure rate but it's little drawback is that it has incomplete gamma function which makes it to be more complicated in expressing its survival function and hazard function. This reason makes Weibull distribution more attractive because of its well defined statistical properties (Gupta and Kundu, 2001). Maxwell introduced Maxwell distribution which is popularly known for modeling data related to science. It can also be used to model positively skewed datasets. But its great failure is incapable to model lifetime dataset which is skewed to the left or right due to its increasing failure rate (Ishaq and Abiodun, 2020). Suppose random variable X follows Maxwell distribution, then its cdf is given as follows;

$$F(x,\beta)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{x^2}{2\beta^2}\right),\beta>0,x\geq 0 \tag{1}$$

where $\gamma\left(\frac{3}{2},\frac{x^2}{2\beta^2}\right)$ is incomplete gamma function and is defined as for $x>0$ and $a>0$, $\gamma(a,x)=\int_0^x t^{a-1}e^{-t}dt$

and its corresponding probability density function is defined by;

$$f(x,\beta)=\sqrt{\frac{2}{\pi}}\frac{x^2e^{-\frac{x^2}{2\beta^2}}}{\beta^3},\beta>0,x\geq 0 \tag{2}$$

where β is a scale parameter.

Exponential distribution is receiving considerable attention, because many scientists used it as their baseline distribution while others remodel it to another distribution. For example, Gupta introduced Exponentiated Exponential distribution (Gupta and Kundu, 2001), it consists of raising power to the cumulative function of exponential distribution by positive parameter which make it to two parameters distribution. Its shape parameter enables it to possess increasing or decreasing failure rate. Oguntunde extended Exponential distribution and proposed Weibull – Exponential distribution (Oguntunde et al., 2015).

Ishaq proposed Maxwell generalized family of distributions (Ishaq and Abiodun, 2020). The Maxwell generalized family is later extended to Maxwell – Weibull distribution. The cumulative distribution and probability density functions of Maxwell generalized family of distributions is respectively given as;

$$F(x,\lambda,\beta)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\beta^2}\left(\frac{M(x,\xi)}{1-m(x,\xi)}\right)^2\right),\beta>0,x\in\mathbb{R} \tag{3}$$

$$F(x,\lambda,\beta)=\frac{2m(x,\xi)}{\beta^2\sqrt{2\pi}\left(1-M(x,\xi)\right)^2}\left(\frac{M(x,\xi)}{1-m(x,\xi)}\right)^2\exp\left(-\frac{1}{2\beta^2}\left(\frac{M(x,\xi)}{1-m(x,\xi)}\right)^2\right) \tag{4}$$

where $M(x,\xi)$, $m(x,\xi)$ and ξ are the cumulative distribution function, probability density function and parameter of the baseline distribution respectively. A new class of Exponential distribution is introduced by applying Maxwell generalized family of distributions.

The method used in proposing the Maxwell generalized family of distribution was introduced by Alzaatreh (Alzaatreh et al., 2013). The cumulative distribution function of a new generalized family of distribution is given by;

$$F(x)=\int_{c}^{N(M(x))} f(x)dx \quad R\subset[c,k]; \quad -\infty\le c<k\le\infty$$

where $N(M(x))$ is the link function of distribution function $M(x)$ for any random variable X and $f(x)$ is the density function of random variable R.

2. The Cumulative Distribution and Probability Density Functions of Maxwell – Exponential Distribution

The section introduces the Maxwell – Exponential (M_{wl}-E) distribution. The motivation behind this research is to obtain flexible distribution that can provide different shapes of the density and hazard functions, and also to provide a great flexibility when tested against its competing models from well-known family of distributions.

The cumulative distribution and probability density functions of M_{wl}-E distribution can be obtained by inserting the cumulative distribution and probability density functions of Exponential distribution in to equations (3) and (4) respectively.

$$F(x,\lambda,\beta)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\beta^2}\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2\right),\lambda,\beta,x>0 \tag{5}$$

$$f(x,\lambda,\beta)=\frac{2\lambda e^{-\lambda x}}{\beta^2\sqrt{2\pi}\left(e^{-\lambda x}\right)^2}\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2\exp\left(-\frac{1}{2\beta^2}\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2\right) \tag{6}$$

where β and λ are the scale parameters. For more simplicity let the cumulative density function be written as $F(x)=F(x,\lambda,\beta)$ and the probability density function $f(x)=f(x,\lambda,\beta)$ throughout the paper.

It can be observed from the figure 1 below, the plot of Maxwell – Exponential distribution possess right and left skewed pattern. Therefore, the Maxwell – Exponential distribution can fit data set that has both right and left-skewed.

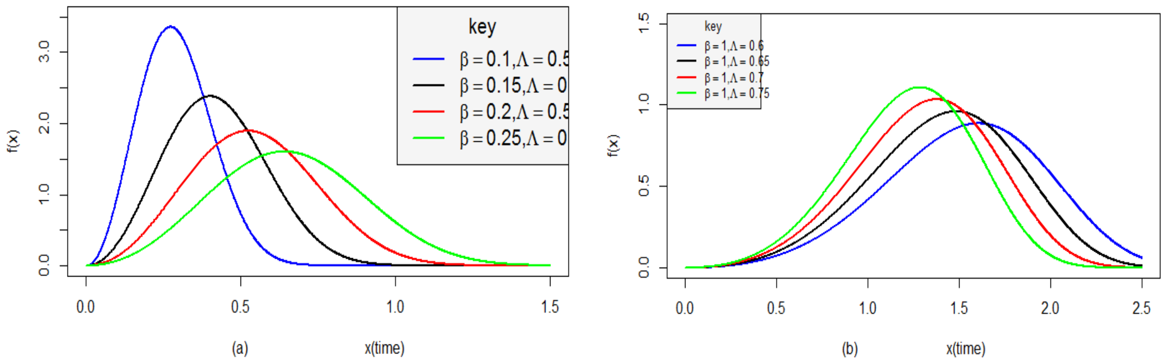

Figure 1. Plot of Probabilty density function of M_{wl}-E distribution at the various values of parameters.

3. An Important Linear Representation

Linear representation of M_{wl}-E density function will be provided here, by applying the Exponential power series expansion to last term of equation (6) for $x > 0$

$$e^{-x} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} x^i$$

Equation (6) now becomes

$$f(x) = \frac{2\lambda}{2^i \beta^{3+2i} \sqrt{2\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} e^{-\lambda x} \frac{\left(1-e^{-\lambda x}\right)^{2+2i}}{\left[1-\left(1-e^{-\lambda x}\right)\right]^{4+2i}} \tag{7}$$

Now consider the expansion for $|z| < 1$, $b > 0$, then the power series is defined to be

$$(1-z)^{-b} = \sum_{j=0}^{\infty} \frac{\Gamma(b+j)}{j!\Gamma(b)} z^j$$

equation (7) can be further rewritten as

$$f(x) = \frac{2\lambda}{2^i \beta^{3+2i} \sqrt{2\pi}} \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!\,j!} \frac{\Gamma(4+2i+j)}{\Gamma(4+2i)} e^{-\lambda x} \left(1-e^{-\lambda x}\right)^{2+2i+j} \tag{8}$$

Now consider the series of generalized binomial expansion

$$(1-z)^{b-1} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(b)}{k!\Gamma(b-k)} z^k \text{ for } |z| < 1, b > 0 \tag{9}$$

Equation (8) can further be simplified by applying (9) as

$$f(x) = \sum_{k=0}^{\infty} D_{(k)} e^{-\lambda(1+k)x} \tag{10}$$

where $D_{(k)} = \frac{2\lambda}{2^i \beta^{3+2i} \sqrt{2\pi}} \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!\,j!} \frac{\Gamma(4+2i+j)}{\Gamma(4+2i)} e^{-\lambda x} \binom{2+2i+j}{k}$.

4. The Survival and Hazard Functions of the Maxwell – Exponential Distribution

Here survival and hazard functions of M_{wl}-E distribution was described.

4.1 The Survival Function

The survival function ($S(x)$) For M_{wl}-E distribution were obtained as

$$S(x) = 1 - \frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{1}{2\beta^2}\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2\right) \tag{11}$$

where $\beta > 0$ and $\lambda > 0$ are scale parameters.

4.2 The Hazard Function

The hazard function ($h(x)$) for M_{wl}-E distribution were obtained as

$$h(x) = \frac{\frac{2\lambda e^{-\lambda x}}{\left(e^{-\lambda x}\right)^2}\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2 \exp\left(-\frac{1}{2\beta^2}\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2\right)}{\beta^3\sqrt{2\pi}\left\{1 - \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{1}{2\beta^2}\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2\right)\right\}} \tag{12}$$

It can be observed from figure 2 below that, the hazard function of Maxwell – Exponential distribution has increasing function.

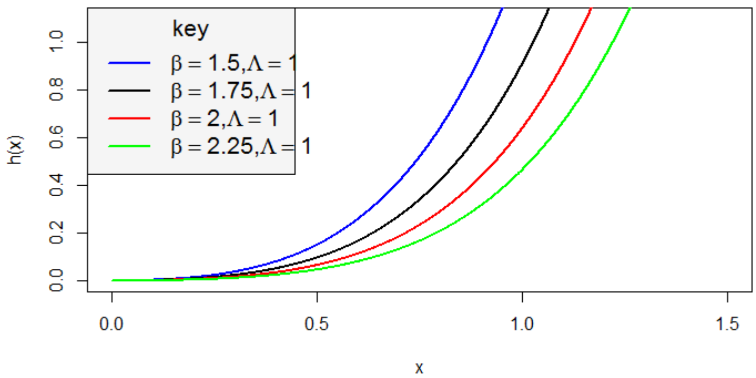

Figure 2. *The Plot of hazard function of M_{wl}-E distribution*

5. The Basic Properties of Maxwell – Exponential Distribution

The basic properties of M_{wl}-E distribution namely; the moments, incomplete moments, probability weighted moments and order statistics are presented here.

5.1 The Moments

Suppose random variable X follows M_{wl}-E distribution, then the non – central moment of X is given by;

$$E(x^r) = \int_{-\infty}^{\infty} x^r f(x)dx \tag{13}$$

Here $f(x)$ is the probability density function of M_{wl}-E distribution, inserting equation (10) into equation (13) gives;

$$E(x^r) = \sum_{k=0}^{\infty} D_{(k)} \int_0^{\infty} x^r e^{-\lambda(1+k)x} dx \tag{14}$$

Let $y = \lambda(1+k)x \Rightarrow x = \dfrac{y}{\lambda(1+k)}$ and $dx = \dfrac{dy}{\lambda(1+k)}$ (15)

By substituting equation (15) into equation (14) we have;

$$E(x^r) = \sum_{k=0}^{\infty} D_{(k)} \int_0^{\infty} \left(\frac{y}{\lambda(1+k)} \right)^r e^{-y} \frac{dy}{\lambda(1+k)}$$

$$= \sum_{k=0}^{\infty} D_{(k)} \frac{1}{\left(\lambda\left(1+k\right)\right)^{r+1}} \int_0^{\infty} y^{r+1-1} e^{-y} dy \tag{16}$$

$$= \sum_{k=0}^{\infty} D_{(k)} \frac{\Gamma(r+1)}{\left(\lambda\left(1+k\right)\right)^{r+1}} \tag{17}$$

Equation (17) gives moments of the M_{wl}-E distribution.

5.2 The Incomplete Moments

The incomplete moments of M_{wl}-E distribution denoted by $\phi_r(t)$ is defined by;

$$\phi_r(t) = \int_{-\infty}^{t} x^r f(x)dx \tag{18}$$

Here $f(x)$ is the pdf of M_{wl}-E distribution defined in equation (10).

$$\phi_r(t) = \sum_{k=0}^{\infty} D_{(k)} \int_0^{t} x^r e^{-\lambda(1+k)x} dx \tag{19}$$

By substituting equation (15) into equation (19) we have;

$$\phi_r(t)=\sum_{k=0}^{\infty}D_{(k)}\int_0^{\lambda(1+k)t}\left(\frac{y}{\lambda(1+k)}\right)^r e^{-y}\frac{dy}{\lambda(1+K)}$$

$$=\sum_{k=0}^{\infty}D_{(k)}\frac{1}{(\lambda(1+k))^{r+1}}\int_0^{\lambda(1+k)t}y^r e^{-y}dy \tag{20}$$

$$=\sum_{k=0}^{\infty}D_{(k)}\frac{1}{(\lambda(1+k))^{r+1}}\gamma(r+1,\lambda(1+k)t) \tag{21}$$

Equation (21) is incomplete moments of the M_{wl}-E distribution.

5.3 The Probability Weighted Moments

The probability weighted moments of $r,q\epsilon R$, and $z=0$ is defined below;

$$\rho(r,s,o)=E\left\{X^sF(X)^r\right\}=\int_{-\infty}^{\infty}x^sF(x)^r f(x)dx \tag{22}$$

Where $F(x)$ and $f(x)$ are given in equation (5) and equation (6) respectively. Thus;

$$F(x)^r f(x)=\lambda\sum_{b,i,j,k,l=0}^{\infty}Q_{(b,i,j,k,l)}e^{-\lambda x}(1-e^{-\lambda x})^{2+2i+3b+2k+l} \tag{23}$$

where $Q_{(b,i,j,k,l)}=\dfrac{(-1)^k B_a(r)A_{b,i}\Gamma(4+2i+3b+2k+l)}{\beta^{3+2i+3b+2k}\,2^{2i+3b+k}\left(\Gamma\left(\frac{3}{2}\right)\right)^b k!l!\Gamma(4+2i+3b+2k)}\sqrt{\dfrac{2}{\pi}}$

$$B_a(r)=\sum_{l=b}^{\infty}(-1)^{l+m}\binom{r}{l}\binom{l}{m}$$

$A_{b,i}=(id_o)^{-1}\sum_{j=1}^{i}(b_j-i+j)d_j$ and $d_j=\dfrac{(-1)^j}{j!(m+j)}$.

Then by substituting equation (23) into equation (22) gives;

$$\rho_{(r,s,o)}=\lambda\sum_{b,i,j,k,l=0}^{\infty}Q_{(b,i,j,k,l)}\int_0^{\infty}x^r e^{-\lambda x}(1-e^{-\lambda x})^{2+2i+3b+2k+l}dx \tag{24}$$

Applying generalized binomial expansion as defined in equation (9) into equation (24) we have;

$$\rho_{(r,s,o)}=\lambda\sum_{b,i,j,k,l,m=0}^{\infty}Q_{(b,i,j,k,l)}(-1)^m\binom{2+2i+3b+2k+l}{m}\int_0^{\infty}x^r e^{-\lambda(1+m)x}dx \tag{25}$$

Let $t=\lambda(1+m)x \Rightarrow x=\dfrac{t}{\lambda(1+m)}$ and $dx=\dfrac{dt}{\lambda(1+m)}$ (26)

By substituting equation (26) into equation (25) gives;

$$\rho_{(r,s,o)}=\lambda\sum_{b,i,j,k,l,m=0}^{\infty}Q_{(b,i,j,k,l)}(-1)^m\binom{2+2i+3b+2k+l}{m}\int_0^{\infty}\left(\frac{t}{\lambda(1+m)}\right)^r\frac{dt}{\lambda(1+m)}$$

$$=\lambda\sum_{b,i,j,k,l,m=0}^{\infty}Q_{(b,i,j,k,l)}(-1)^m\frac{(-1)\Gamma(r+1)}{(\lambda(1+m))^{r+1}}\binom{2+2i+3b+2k+l}{m} \quad (27)$$

Equation (27) is probability weighted moments of M_{wl}-E distribution.

5.4 The Order Statistics

Let $X_k(k=1,\ldots,n)$ denote M_{wl}-E random variable with samples size n, then the density function of the k^{th} order statistics of M_{wl}-E distribution denoted by $f_{k,n}(x)$ is given as;

$$f_{k,n}(x)=\frac{n!}{(k-1)!(n-k)!}f(x)F(x)^{k-1}\left[1-F(x)\right]^{n-1} \quad (28)$$

where $F(x)$ and $f(x)$ are defined in equation (5) and equation (6) respectively.
But

$$F(x)^{k-1}\left[1-F(x)\right]^{n-1}=\sum_{p=0}^{\infty}(-1)^p\binom{n-k}{p}F(x)^{k+p-1} \quad (29)$$

By substituting equation (29) into equation (28) we have;

$$f_{k,n}(x)=\frac{n!}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}(-1)^p\binom{n-k}{p}F(x)^{k+p-1} \quad (30)$$

Now, $f(x)F(x)^{k+p-1}$ is defined for $s=k+p-1$ from equation (23). Thus,

$$f_{k,n}(x)=\lambda\sum_{m,i,j,k,l,p=0}^{\infty}\frac{(-1)^{p+q}n!Q_{(b,i,j,k,l)}}{(k-1)!(n-k)!}\binom{n-k}{p}e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{2+2i+3b+2k+l}$$

$$f_{k,n}(x)=\lambda\sum_{m,i,j,k,l,p=0}^{\infty}\frac{(-1)^{p+q}n!Q_{(b,i,j,k,l)}}{(k-1)!(n-k)!}\binom{n-k}{p}\binom{2+2i+3b+2k+l}{n}e^{-\lambda(1+p)x} \quad (31)$$

6. Parameter Estimation

Let $X_t(t=1,\ldots,n)$ denote a random sample from M_{wl}-E with parameters λ and β. By using the maximum likelihood estimation method, the likelihood function gives;

$$f(x_t(t=1,...n),\lambda,\beta)=\prod_{i=1}^{n}\left[\frac{2\lambda e^{-\lambda x}}{\beta^3\sqrt{2\pi}\left(e^{-\lambda x}\right)^2}\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2\exp\left(-\frac{1}{2\beta^2}\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2\right)\right] \tag{32}$$

Let $l=\log f(x_t(t=1,...n),\lambda,\beta)$ denote the log – likelihood function then,

$$\begin{aligned} l &= n\log(2)+n\log(\lambda)-\lambda\sum_{t=1}^{n}x_t-3n\log(\beta)-\frac{n}{2}\log(2\pi)-2\sum_{t=1}^{n}\log\left[\mathrm{e}^{-\lambda x_t}\right]+ \\ & 2\sum_{t=1}^{n}\log\left[\frac{1-e^{-\lambda x_t}}{e^{-\lambda x_t}}\right]-\frac{1}{2\beta^2}\sum_{t=1}^{n}\left[\frac{1-e^{-\lambda x_t}}{e^{-\lambda x_t}}\right]^2 \end{aligned} \tag{33}$$

The maximum likelihood estimated parameters can be obtained by differentiating l with respect to parameters λ *and* β and equating to zero.

$$\frac{dl}{d\beta}=-\frac{3n}{\beta}+\frac{1}{\beta^3}\sum_{t=1}^{n}\left(\frac{1-e^{-\lambda x_t}}{e^{-\lambda x_t}}\right)^2=0 \tag{34}$$

$$\Rightarrow\ -3n\beta^2+\sum_{t=1}^{n}\left(\frac{1-e^{-\lambda x_t}}{e^{-\lambda x_t}}\right)^2=0$$

$$\Rightarrow\ \hat{\beta}=\sqrt{\frac{1}{3n}\sum_{t=1}^{n}\left(\frac{1-e^{-\lambda x_t}}{e^{-\lambda x_t}}\right)^2} \tag{35}$$

Equation (35) is the maximum likelihood estimate of the parameter β

$$\frac{dl}{d\lambda}=\frac{n}{\lambda}+\sum_{t=1}^{n}x_t+2\sum_{t=1}^{n}\left(\frac{x_t}{1-e^{-\lambda x_t}}\right)-\frac{1}{\beta^2}\sum_{t=1}^{n}\left(\frac{1-e^{-\lambda x_t}}{e^{-\lambda x_t}}\right)\left(\frac{x_t}{e^{-\lambda x_t}}\right)=0 \tag{36}$$

Substituting equation (35) into (36) and solving the equation will yield the maximum likelihood estimate of the parameter λ, statistical software could be used to solve the equation.

7. Applications

This section described a simulation study using quantile function and application to real – life datasets to demonstrate the flexibility of M_{wl}-E distribution.

7.1 The Quantile Function

We obtained the quantile function of M_{wl}-E distribution by inverting its cumulative distribution function as given in equation (5).

$$F(x,\lambda,\beta)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\beta^2}\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2\right)$$

The cumulative distribution function can be express as;

$$F(x,\lambda,\beta)=\frac{\gamma(n,m)}{\Gamma(n)} \tag{37}$$

where $n=\frac{3}{2}$ and $m=\frac{1}{2\beta^2}\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2$ (38)

The quantile function of the proposed M_{wl}-E model can be obtained from equation (37) below (Oluyede B, 2018).

$$m=\gamma^{-1}\left(n,u\Gamma(n)\right) \tag{39}$$

By substituting for n and m equation (39) becomes;

$$\frac{1}{2\beta^2}\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2=\gamma^{-1}\left(\frac{3}{2},u\Gamma\left(\frac{3}{2}\right)\right) \tag{40}$$

This can be written as;

$$\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)=\left\{2\beta^2\gamma^{-1}\left(\frac{3}{2},u\Gamma\left(\frac{3}{2}\right)\right)\right\}^{\frac{1}{2}} \tag{41}$$

This can be express as,

$$\frac{w}{1-w}=z \tag{42}$$

where $w=1-e^{-\lambda x}$ and $z=\left\{2\beta^2\gamma^{-1}\left(\frac{3}{2},u\Gamma\left(\frac{3}{2}\right)\right)\right\}^{\frac{1}{2}}$ (43)

Now w is given as;

$$w=\frac{z}{1+z} \tag{44}$$

Substituting w in equation (44) becomes;

$$1-e^{-\lambda x}=\frac{z}{1+z} \tag{45}$$

$$e^{-\lambda x}=1-\left(\frac{z}{1+z}\right) \tag{46}$$

Taking $\log$ to both sides of the equation gives;

$$\lambda x=\log\left(1-\left(\frac{z}{1+z}\right)\right) \tag{47}$$

The quantile function of the proposed M_{wl}-E distribution is given as;

$$x=-\frac{1}{\lambda}\log\left(1-\left(\frac{z}{1+z}\right)\right) \tag{48}$$

$$x=-\frac{1}{\lambda}\log\left(1-\left(\frac{\left\{2\beta^2\gamma^{-1}\left(\frac{3}{2},u\Gamma\left(\frac{3}{2}\right)\right)\right\}^{\frac{1}{2}}}{1+\left\{2\beta^2\gamma^{-1}\left(\frac{3}{2},u\Gamma\left(\frac{3}{2}\right)\right)\right\}^{\frac{1}{2}}}\right)\right) \tag{49}$$

Equation (49) is the quantile of the proposed M_{wl}-E distribution.

7.2 Simulation

The simulation of the random numbers is based on the quantile function defined in equation (36), data were generated for different sample size (n) starting from 20, 50, 80, 120, 320, 720 and 1120 at different values of parameters, by fixing $\beta = 3.5$ and $\lambda = 5.2$. After repeating the simulation 1000 times, the mean (M), variance (Var) and mean squared error (MSE) was obtained. It can be observed that in Table (1). The mean estimate for each parameter is approaching its fixed value.

Table 1. The performance of the (M_{wl}-E) distribution simulation

Sample Size (n)	Mean (M)	Variance (Var)	MSE
20	$\beta = 5.594546$	23.954450	28.341570
	$\lambda = 5.765681$	2.3213180	2.6413140
50	$\beta = 4.112448$	3.3493340	3.7244270
	$\lambda = 5.398237$	0.8226398	0.8619376
80	$\beta = 3.874757$	1.6565820	1.7970240
	$\lambda = 5.335095$	0.4208483	0.4890990
120	$\beta = 3.760524$	1.0262830	1.0941560
	$\lambda = 5.299490$	0.3165021	0.3264000
320	$\beta = 3.605025$	0.3470734	0.3581035
	$\lambda = 5.241440$	0.1195623	0.1212795
720	$\beta = 3.544303$	0.1396096	0.1415724
	$\lambda = 5.217804$	0.0514281	0.0517451
1120	$\beta = 3.532054$	0.0872496	0.0882771
	$\lambda = 5.213228$	0.0322500	0.0324250

7.3 Model Comparison

The proposed M_{wl}-E distribution will be compared with other distributions using real – life datasets to demonstrate its flexibility. The competing models are; Exponential distribution, Exponentiated – Exponential distribution and Burr X – Exponential distribution. The competing models will be represented as E_{xp}-X, E_{xp}-E and B_{ux}-E respectively. The first competing model has only scale parameter. It is also the baseline distribution of the proposed model research. The second and third models have two parameters (scale and shape parameters). The distribution functions of the competitive models are respectively given as;

$$f(x,\lambda)=\lambda e^{-\lambda x}, x,\lambda>0 \tag{50}$$

$$f(x,\alpha,\lambda)=\alpha\lambda\left[1-e^{-\lambda x}\right]^{\alpha-1}e^{-\lambda x}, x,\alpha,\lambda>0 \tag{51}$$

$$f(x,\alpha,\lambda)=2\alpha\lambda\left(\frac{1-e^{-\lambda x}}{\left(e^{-\lambda x}\right)^2}\right)\exp\left[-\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2\right]\left\{1-\exp\left[-\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^2\right]\right\}^{\alpha-1}, x,\alpha,\lambda>0 \tag{52}$$

To assess the best model from the competitive distributions we use some information criteria, this includes; Akaike Information Criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike Information Criterion (AIC_C) and Hannan – Quinn Information Criterion (HQIC). The least value of these criteria implies the best among the distributions that fit data sets, the formulae of these criteria are given as follow;

$$AIC=2k-2\ln(L) \tag{53}$$

$$BIC=2k\ln(N)-2\ln(L) \tag{54}$$

$$AIC_C=AIC+\frac{2k^2+2k}{N-k-1} \tag{55}$$

$$HQIC=2k\ln(\ln(N)) \tag{56}$$

where L is the value of the likelihood, N is the number of recorded measurement and k is the number of estimated parameter.

7.4 First Data Set

The first dataset is strengths of 1.5cm glass fiber. It was used previously by (Oguntunde et al., 2017)

Table 2. Strengths of 1.5cm glass fiber

0.55	0.74	0.77	0.81	0.84	1.24	0.93	1.04	1.11	1.13	1.30	1.25	1.27
1.28	1.29	1.48	1.36	1.39	1.42	1.48	1.51	1.49	1.49	1.50	1.50	1.55
1.52	1.53	1.54	1.55	1.61	1.58	1.59	1.60	1.61	1.63	1.61	1.61	1.62
1.62	1.67	1.64	1.66	1.66	1.66	1.70	1.68	1.69	1.70	1.78	1.73	1.76
1.76	1.77	1.89	1.81	1.82	1.84	1.84	2.00	2.01	2.24			

Table 3. The information criteria on the strengths of 1.5cm glass fiber

Models	Estimates	AIC	BIC	AIC_C	HQIC	p-value
E_{xp}-X	$\lambda = 0.6648$	176.611	178.738	176.677	177.446	9.278e-10
E_{xp}-E	$\lambda = 0.8693$ $\alpha = 1.9726$	140.457	144.712	140.661	142.128	3.204e-05
B_{ux}-E	$\lambda = 0.5350$ $\alpha = 1.9286$	48.2448	52.4990	48.4482	49.9142	1.376e-05
M_{wl}-E	$\beta = 1.8554$ $\lambda = 0.9043$	37.1853	41.4396	37.3887	38.8556	0.01646

7.5 Second Data Set

The second data is the strengths of glass fibers data set comprising 63 observations. It was used by (Ishaq and Abiodun 2020).

Table 4. The strengths of glass fibers

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68
1.73	1.81	2.00	0.74	1.04	1.27	1.39	1.49	1.53	1.59
1.61	1.66	1.68	1.76	1.82	2.01	0.77	1.11	1.28	1.42
1.50	1.50	1.54	1.60	1.62	1.66	1.69	1.76	1.84	2.24
0.81	1.13	1.29	1.48	1.50	1.55	1.61	1.62	1.66	1.70
1.77	1.84	0.84	1.24	1.30	1.48	1.51	1.55	1.61	1.63
1.67	1.70	1.78	1.89						

Table 5. The information criteria on the strengths of glass fibers

Models	Estimates	AIC	BIC	AIC_C	HQIC	p-value
E_{xp}-X	$\lambda = 0.6637$	179.661	181.804	179.726	180.504	5.497e-10
E_{xp}-E	$\lambda = 0.8693$ $\alpha = 1.9726$	142.814	147.099	143.014	144.499	2.63e-05
B_{ux}-E	$\lambda = 0.4793$ $\alpha = 1.9710$	48.4701	52.7563	48.6701	50.1559	0.02577
M_{wl}-E	$\beta = 1.7950$ $\lambda = 0.9037$	37.7465	42.0328	37.9465	39.4323	0.00258

It can be observed from tables (3) and (5) that the M_{wl}-E distribution is more flexible compare to other competing models because it has the least value of information criteria.

8. Research Findings

An extension of exponential distribution namely Maxwell – Exponential distribution was proposed by using Maxwell generalized family of distribution. Its statistical properties such as moments, incomplete moments, probability weighted moments and order statistics was discussed. Simulation study was conducted to demonstrate the potentiality of the proposed model estimator. The proposed model was applied to real life data sets, in both data sets it was shown a great flexibility over the competing models.

9. Conclusion

A new extension of Exponential distribution called Maxwell – Exponential distribution was proposed using Maxwell generalized family of distributions. Its statistical properties have been discussed. The plot of Maxwell – Exponential density function possesses the pattern of right and left skewed at different values of parameters. The parameters of the study model were estimated using the maximum likelihood method. The Simulation study was carried out to demonstrate the potentiality of proposed model parameters. An application of Maxwell – Exponential distribution to the real – life data sets was carried out. In both given datasets, the proposed model has performed well more than to the other competing models. Therefore the proposed model is a great substitute to the competing models in modeling these datasets concerning the first data (strengths of 1.5cm glass fiber) and the second data (The strengths of glass fibers).

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11. References

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