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The Extended Odd Fréchet Inverted Exponential Distribution: Theoretical Properties and Application

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Abstract

In this paper, a novel three-parameter model called the extended odd Fréchet inverted exponential distribution is proposed. The density function can be expressed as a linear mixture of generalized inverse exponential densities and can be skewed to the right. Moreso, the hazard rate function of the novel model can be unimodal, increasing-decreasing and inverted bathtub shaped. Some reliability and statistical properties are derived and the maximum likelihood estimation technique described for the proposed novel model. The Monte Carlo simulations are performed to assess the performance of the maximum likelihood estimator for both small and large sample, and the numerical results show that as the sample size increases, the mean estimates tend towards the parameters true values with minimum errors. The performance of the novel model is illustrated by means of two real-life datasets and both real-life datasets show that the novel model offers a better fit compared to some other competing models.

Keywords: Inverted bathtub failure rate, Inverse exponential distribution, Maximum likelihood estimation, Skewed datasets, Simulation.

1. Introduction

The interest of statisticians and data scientists have been in creating novel univariate probability models by adding one or more shape parameter(s) to well-known lifetime models in order to create new efficient models that offer superior flexibility in modeling datasets in numerous areas such as financial analysis, medicine, life testing, econometric, and engineering, among others. The significant role of these added shape parameters is to vary the model tail-weights, introduce skewness, and provide greater ability to model both non-monotonic and monotonic failure rates. The analytically tractable inverted exponential distribution (IED) considered as a modification of the exponential distribution have been widely utilized for investigating various datasets given its lack of memory property. Also, the IED is viewed as a distinct case of the inverted Weibull distribution proficient for modelling real-life progressions with upside down bathtub failure hazard rate. However, the applicability of the IED is restricted given that it exhibits a decreasing density function and nonconstant failure rate thus cannot be used in analysing lifetime datasets with bathtub shaped failure rate form.

Recently, several authors have attempted to improve the flexibility, practicability and modelling capability of the IED by introducing various generalizations and extensions using different families of distributions. For example, the Beta IED studied by Singh and Goel (2015), Kumaraswamy IED introduced by Oguntunde *et al.* (2017), Alpha power IED proposed by Unal *et al.* (2018), Gompertz IED proposed by Oguntunde *et al.* (2018), Weibull Alpha power IED proposed by Efe-Eyefia *et al.* (2019), odd Fréchet IED proposed by Sharifah (2019), Gompertz Alpha power IED introduced by Eghwerido *et al* (2019), Kumaraswamy Alpha power IED pioneered by Zelibe *et al.* (2019), exponentiated exponential IED studied by Asongo *et al.* (2020), extended generalized IED introduced by Bashiru (2021), Exponentiated odd Lindley IED studied by Eraikhuement *et al.* (2021), Weibull exponential IED studied by Chama *et al.* (2021), generalized IED introduced by Al-Omari *et al*. (2021), odd Lomax IED introduced by Leren *et al.* (2021) and Exponential Gompertz IED proposed by Adubisi and Adubisi (2022).

The aim of this article is to propose and study a new three-parameter model termed the extended odd Fréchet inverse exponential (EOF_{IE}) distribution, which has numerous desirable properties. The cdf of the EOF_{IE} model is given by

$$F\left(x;\theta,\alpha,\beta\right)=e^{-\left(\left(e^{-\frac{\beta}{x}}\right)^{-\alpha}-1\right)^{\theta}}, \qquad (1)$$

for all $x>0,\ \theta>0,\ \alpha>0,$ and $\beta>0$. The EOF_{IE} model is very flexible, capable of modelling positive real-life datasets, and offers a better fit than six competing existing generalization of the IED by means of two applications. The IED has the cumulative distribution function (cdf) and probability density function (PDF), which are given by

$$G_{IE}(x;\beta)=e^{-\frac{\beta}{x}} \text{ and } g_{IE}(x;\beta)=\frac{\beta}{x^2}e^{-\frac{\beta}{x}}, \qquad (2)$$

respectively, where $\beta>0$ is positive scale parameter.

The rest of this article is structured as follows. In Unit 2, the EOF_{IE} distribution is defined. In Unit 3, the quantile function, median, 25th, and 75th percentiles are derived. In Unit 4, a very useful representation for the EOF_{IE} density and distribution functions are derived. Basic theoretical properties of the EOF_{IE} are derived in Unit 5. In Unit 6, the EOF_{IE} parameters are estimated through ML estimation technique. Parameters of the EOF_{IE} distribution is also estimated in Unit 7 through simulation studies. In Unit 8, the EOF_{IE} distribution flexibility is explored through the application of two real-life datasets. Finally, Unit 9 concludes the article.

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2. The EOF$_{IE}$ model

Recently, Nasiru (2018) introduced a new family of distributions called the extended odd Fréchet-G (EOF-G) family with two extra shape parameters. The CDF of the EOF-G family is given by

$$F\left(x;\alpha,\theta,\psi\right)=\int_0^{\frac{G(x;\psi)^{\alpha}}{\left[1-G(x;\psi)^{\alpha}\right]}}\left[\theta x^{-\theta-1}e^{-x^{-\theta}}\right]dx \quad =e^{-\left(\frac{1-G(x;\psi)^{\alpha}}{G(x;\psi)^{\alpha}}\right)^{\theta}}, \tag{3}$$

where $G\left(x;\psi\right)$ is the baseline cdf with a vector of unknown parameters ψ, $\theta>0$, and $\alpha>0$. The resultant pdf of (3) takes the form

$$f\left(x;\alpha,\theta,\psi\right)=\frac{\alpha\theta g\left(x;\psi\right)\left[1-G\left(x;\psi\right)^{\alpha}\right]^{\theta-1}}{\left[G\left(x;\psi\right)\right]^{\alpha\theta+1}}e^{-\left(\frac{1-G(x;\psi)^{\alpha}}{G(x;\psi)^{\alpha}}\right)^{\theta}}, \tag{4}$$

where $g\left(x;\psi\right)$ is the baseline pdf with a vector of unknown parameters ψ. From now onward, we will denote X a random variable having the pdf (4) as $X \sim EOF-G\left(\theta,\alpha,\psi\right)$.

The hazard rate function (hrf) of the EOF-G family is given by

$$h\left(x;\alpha,\theta,\psi\right)=\frac{\alpha\theta g\left(x;\psi\right)\left[1-G\left(x;\psi\right)^{\alpha}\right]^{\theta-1}}{\left[G\left(x;\psi\right)\right]^{\alpha\theta+1}\left(1-e^{-\left(\frac{1-G(x;\psi)^{\alpha}}{G(x;\psi)^{\alpha}}\right)^{\theta}}\right)}e^{-\left(\frac{1-G(x;\psi)^{\alpha}}{G(x;\psi)^{\alpha}}\right)^{\theta}}. \tag{5}$$

Considering the cdf and pdf of the IED (2). Hence, the pdf of the EOF$_{IE}$ with the set of parameters $\xi=\left(\alpha,\theta,\beta\right)$ corresponding to (4) is expressed as

$$f\left(x;\xi\right)=\frac{\alpha\theta\left(\frac{\beta}{x^2}e^{-\frac{\beta}{x}}\right)\left[1-\left(e^{-\frac{\beta}{x}}\right)^{\alpha}\right]^{\theta-1}}{\left(e^{-\frac{\beta}{x}}\right)^{\alpha\theta+1}}e^{-\left(\left(e^{-\frac{\beta}{x}}\right)^{-\alpha}-1\right)^{\theta}}, \tag{6}$$

where $\beta>0$ is a scale parameter, and $\theta>0$ and $\alpha>0$ are shape parameters. The random variable X having the pdf (6) is denoted by $X \sim EOF_{IE}\left(\theta,\alpha,\beta\right)$.

The survival function (sf) of the EOF$_{IE}$ is defined as

$$R_{SF}\left(x;\xi\right)=1-e^{-\left(\left(e^{-\frac{\beta}{x}}\right)^{-\alpha}-1\right)^{\theta}}, \tag{7}$$

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The hazard rate function (hrf) and reversed hrf (rhrf) of the EOF$_{IE}$ are defined as

$$h(x;\xi)=\frac{\alpha\theta\left(\frac{\beta}{x^{2}}e^{-\frac{\beta}{x}}\right)\left[1-\left(e^{-\frac{\beta}{x}}\right)^{\alpha}\right]^{\theta-1}}{\left(e^{-\frac{\beta}{x}}\right)^{\alpha\theta+1}\left(1-e^{-\left(\left(e^{-\frac{\beta}{x}}\right)^{-\alpha}-1\right)^{\theta}}\right)}e^{-\left(\left(e^{-\frac{\beta}{x}}\right)^{-\alpha}-1\right)^{\theta}},\text{ and } r(x;\xi)=\frac{\alpha\theta\left(\frac{\beta}{x^{2}}e^{-\frac{\beta}{x}}\right)\left[1-\left(e^{-\frac{\beta}{x}}\right)^{\alpha}\right]^{\theta-1}}{\left(e^{-\frac{\beta}{x}}\right)^{\alpha\theta+1}}$$

The odds function (of) and cumulative hrf (chrf) of the EOF$_{IE}$ are defined as

$$O(x;\xi)=\frac{e^{-\left(\left(e^{-\frac{\beta}{x}}\right)^{-\alpha}-1\right)^{\theta}}}{1-e^{-\left(\left(e^{-\frac{\beta}{x}}\right)^{-\alpha}-1\right)^{\theta}}},\text{ and } \mathrm{H}(x;\xi)=-\ln\left[1-e^{-\left(\left(e^{-\frac{\beta}{x}}\right)^{-\alpha}-1\right)^{\theta}}\right].$$

Figure 1 shows some possible shapes of the pdf and hrf of the EOF$_{IE}$ distribution for selected model parameters values. It is observed from the PDF plots that the EOF$_{IE}$ model can be right skewed and unimodal. The hrf of the EOF$_{IE}$ model has the merit of being capable of modeling increasing, decreasing and inverted bathtub shape forms for increment value of the EOF$_{IE}$ parameters.

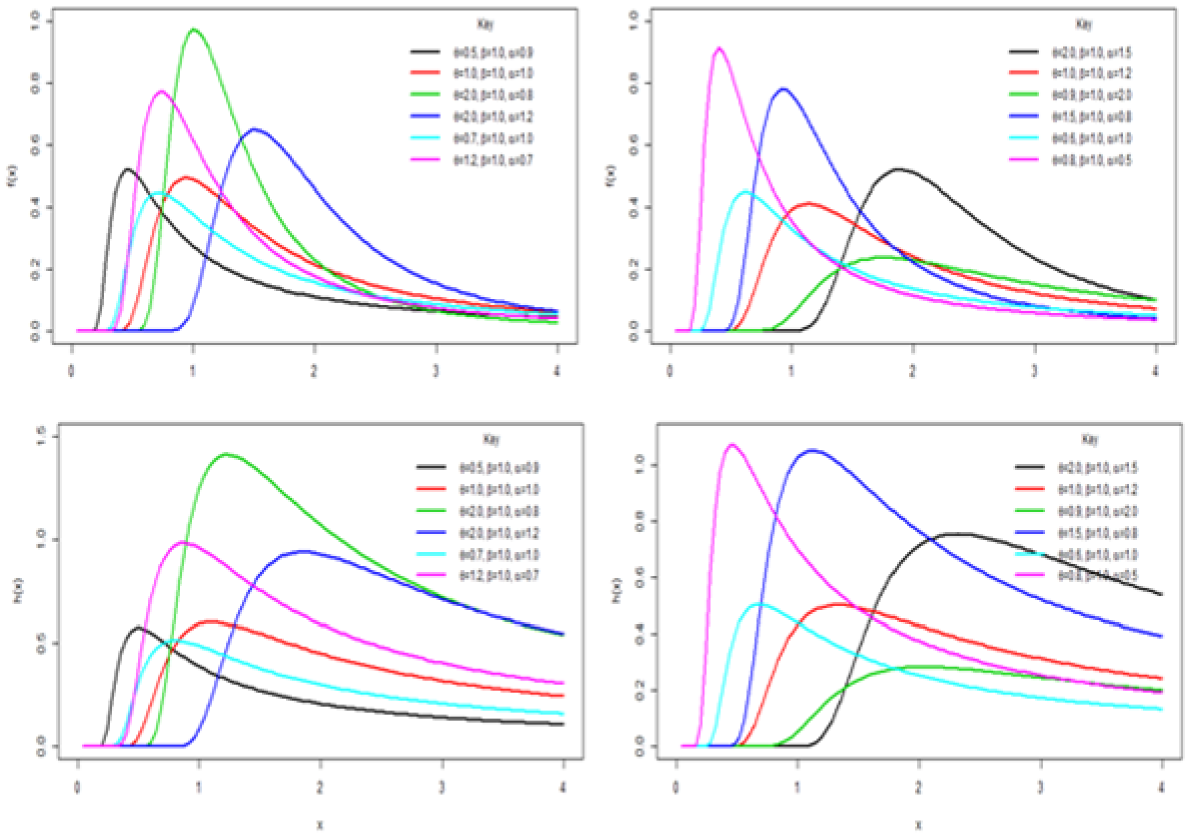

Figure 1: Plots of the pdf ***(top panels)*** and hrf ***(bottom panels)*** of the EOF$_{IE}$ model for selected values of α, θ and β.

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3. Quantile function

Let X denote a random variable such that $X \sim EOF_{IE}(\xi)$ with $\xi = (\alpha, \theta, \beta)$. The quantile function is derived by inverting the cdf of any distribution based on the uniform distribution. Moreso, the uniform random variables are effortlessly spawned in most statistical softwares, the quantile function is considered valuable for simulating random values from the EOF_{IE} distribution. The quantile function $Q(u)$ for $u \in (0,1)$ is obtained by inverting (1) as

$$Q(u) = -\frac{\beta}{\log\left[\left(1+\left(-\log(u)\right)^{\frac{1}{\theta}}\right)^{-\frac{1}{\alpha}}\right]}, \qquad u \in (0,1) \tag{8}$$

The median (M), 25th and 75th percentiles of X are attained by putting $u = 0.5$, $u = 0.25$ and $u = 0.75$ in (8), correspondingly.

$$M = -\frac{\beta}{\log\left[\left(1+\left(-\log(0.5)\right)^{\frac{1}{\theta}}\right)^{-\frac{1}{\alpha}}\right]}, \qquad u \in (0,1)$$

$$Q(0.25) = -\frac{\beta}{\log\left[\left(1+\left(-\log(0.25)\right)^{\frac{1}{\theta}}\right)^{-\frac{1}{\alpha}}\right]}, \qquad u \in (0,1)$$

$$Q(0.75) = -\frac{\beta}{\log\left[\left(1+\left(-\log(0.75)\right)^{\frac{1}{\theta}}\right)^{-\frac{1}{\alpha}}\right]}, \qquad u \in (0,1).$$

Then, the quantile function of EOF_{IE} density function obtained by $\frac{dQ(u)}{du}$, is expressed as

$$q(u) = -\frac{\beta\left(-\log(u)\right)^{\frac{1}{\theta}}}{\alpha \times \theta \times u \log(u) \times \log\left[\left(1+\left(-\log(u)\right)^{\frac{1}{\theta}}\right)^{-\frac{1}{\alpha}}\right]^2 \times \left(1+\left(-\log(u)\right)^{\frac{1}{\theta}}\right)}, \qquad u \in (0,1) \tag{9}$$

3.1 Quantile approach for skewness and kurtosis

The quantile methodology of valuing the skewness and kurtosis of a distribution is chiefly useful when the distribution subsists in a closed form or in a simple analytical system. Kenney and Keeping (1962), and Moore (1988) suggested a quantile measure-based method for skewness and kurtosis, respectively. The Bowley skewness and Moor's kurtosis for EOF_{IE} can be examined by using the quantile function (8) as follows:

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$$S_k = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \text{ and } K_s = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \qquad (10)$$

where $Q(.)$ is the EOF$_{IE}$ quantile function. The Bowley skewness and Moor's kurtosis measures do not hinge on the moments of the distribution and are almost unresponsive to outliers. Numerical values for the median (M), 25th and 75th percentiles, skewness, kurtosis, and interquartile range (IQR) for some nominated parameter values are provided in Table 1. Numerical values printed in Table 1 shows that as θ increases cross different parameter values of α and β, the skewness and kurtosis decrease while the median, IQR, 25th, and 75th percentiles increase. Also, the skewness and kurtosis remain constant within the parameter values of θ for different parameter values of α while β is held constant. Table 1 presents the Median (M), 25th and 75th percentiles, skewness (Sk), kurtosis (Ks) and interquartile range (IQR) for different parameter values.

Table 1: Descriptive statistics of the EOF$_{IE}$ model

θ	α	β	**M**	**25th**	**75th**	**Sk**	**Ks**	**IQR**
0.5	0.6	1.0	1.529	0.560	7.546	0.7222	4.231	6.986
0.5	1.2	1.0	3.058	1.119	15.092	0.722	4.231	13.972
0.5	2.0	1.0	5.097	1.865	25.153	0.722	4.231	23.287
1.0	0.5	1.0	0.949	0.575	1.977	0.466	1.568	1.403
1.0	1.5	1.0	2.848	1.725	5.932	0.466	1.568	4.208
1.0	2.0	1.0	3.798	2.299	7.910	0.466	1.568	5.610
1.5	1.2	1.0	2.075	1.485	3.317	0.357	1.024	1.832
1.5	1.6	1.0	2.766	1.980	4.423	0.357	1.024	2.443
1.5	2.2	1.0	3.803	2.723	6.082	0.357	1.024	3.359

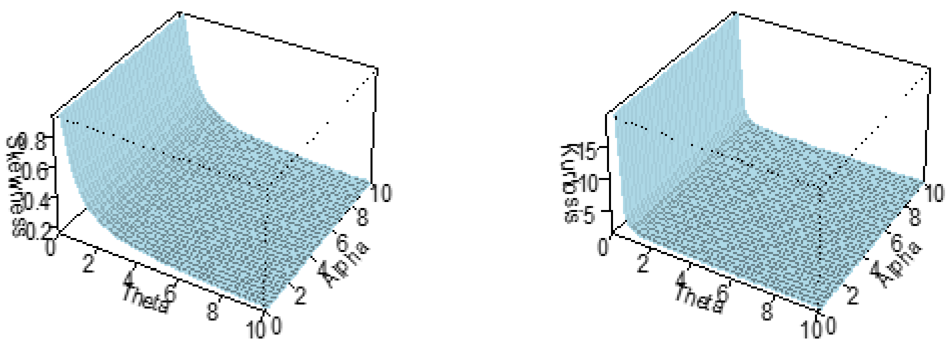

Figure 2: Bowley's skewness and Moor's kurtosis plots $(\beta = 1)$ of the EOF$_{IE}$ distribution.

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Figures 2 depict the three-dimensional plots of the skewness and kurtosis measures. It is clear that the skewness and kurtosis values are increasing in α and decreasing in θ and independent of β which corresponds with the skewness and kurtosis results in Table 1.

4. Mixture representations of the EOF$_{IE}$ distribution

Mixture representation is useful when deriving the statistical properties of a novel model. The series expanded form is easier to work with when presenting the statistical characteristics of the proposed novel model. The EOF$_{IE}$ distribution PDF and CDF mixture representations are derived below.

4.1 Expansion of the pdf of the EOF$_{IE}$ distribution

The series expansion is carried out using Taylor series expansion (TSE), the EOF$_{IE}$ distribution PDF is expressed as

$$f(x)=\frac{\alpha\theta\beta}{x^2}\sum_{i=0}^{\infty}\frac{(-1)^i e^{-\beta/x}\left(1-\left(e^{-\beta/x}\right)^{\alpha}\right)^{\theta(i+1)-1}}{i!\left(e^{-\beta/x}\right)^{\alpha\theta(i+1)+1}} \tag{11}$$

Also (11) can be expressed as

$$f(x)=\frac{\alpha\theta\beta}{x^2}\sum_{i=0}^{\infty}\frac{(-1)^i e^{-\beta/x}\left(1-\left(e^{-\beta/x}\right)^{\alpha}\right)^{\theta(i+1)-1}\left(1-\left[1-e^{-\beta/x}\right]\right)^{-\{\alpha\theta(i+1)+1\}}}{i!} \tag{12}$$

The application of generalized binomial series expansion (BSE) to (12) gives

$$f(x)=\frac{\alpha\theta\beta}{x^2}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\binom{\alpha\theta(i+1)+j}{j}e^{-\beta/x}\left(1-\left(e^{-\beta/x}\right)^{\alpha}\right)^{\theta(i+1)-1}\left(1-e^{-\beta/x}\right)^{j} \tag{13}$$

Now, applying the BSE, $(1-z)^{\lambda-1}=\sum_{i=0}^{\infty}(-1)^j\binom{\lambda-1}{j}z^j,\quad |z|<1$, three times in (13) yields

$$f(x)=\frac{\alpha\theta\beta}{x^2}\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}w_{i,j,k,m,q}\left(e^{-\beta/x}\right)^{q+1} \tag{14}$$

where $w_{i,j,k,m,q}=\frac{(-1)^{i+k+m+q}}{i!}\binom{\alpha\theta(i+1)+j}{j}\binom{\theta(i+1)-1}{k}\binom{\alpha k}{m}\binom{m+j}{q}$. Hence, the mixture representation of EOF$_{IE}$ density function is obtained as

$$f(x)=\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\frac{\vartheta_{i,j,k,m,q}}{x^2}e^{-\frac{\beta(q+1)}{x}} \tag{15}$$

where $\vartheta_{i,j,k,m,q}=\alpha\theta\beta w_{i,j,k,m,q}$.

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4.2 Expansion of the cdf of the EOF$_{IE}$ distribution

Using TSE, the EOF$_{IE}$ distribution CDF is expressed as

$$F(x)=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{1-\left(e^{-\beta/x}\right)^{\alpha}}{\left(e^{-\beta/x}\right)^{\alpha}}\right)^{\theta i} \tag{16}$$

Equation (16) can be reparametrized as

$$F(x)=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{1-\left(e^{-\beta/x}\right)^{\alpha}}{1-\left[1-\left(e^{-\beta/x}\right)^{\alpha}\right]}\right)^{\theta i} \tag{17}$$

The application of generalized BSE to (17) gives

$$F(x)=\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}\binom{\theta i+j-1}{j}\left(1-\left(e^{-\beta/x}\right)^{\alpha}\right)^{\theta i+j} \tag{18}$$

Now, applying the BSE, $(1-z)^{\lambda-1}=\sum_{i=0}^{\infty}(-1)^j\binom{\lambda-1}{j}z^j,\quad |z|<1$, two times in (18), the CDF is given as

$$F(x)=\sum_{i,j,k,l=0}^{\infty}\vartheta_{i,j,k,l}e^{-\frac{\beta l}{x}} \tag{19}$$

where $\vartheta_{i,j,k,l}=\frac{(-1)^{i+k}}{i!}\binom{\theta i+j-1}{j}\binom{\theta i+j}{k}\binom{\alpha k}{l}$. Furthermore, the expansion of $\left[F(x)\right]^s$, for an integer *s* is given as

$$\left[F(x)\right]^s=\sum_{p=0}^{s}\sum_{l,t,u,s=0}^{\infty}\psi_{p,l,t,u,s}e^{-\frac{\beta s}{x}} \tag{20}$$

where $\psi_{p,l,t,u,s}=\frac{(-1)^{l+u}(p)^l}{i!}\binom{s}{p}\binom{\theta l+t-1}{t}\binom{\theta l+t}{u}\binom{\alpha u}{s}$.

5. Statistical properties of the EOF$_{IE}$ distribution

This part presents some necessary statistical properties of the EOF$_{IE}$ distribution. These include the mode, ordinary and incomplete moments, moment generating function, Bonferroni and Lorenz curves, Shannon and Rényi entropies, probability weighted moments, order statistics and stress-strength reliability.

5.1 Mode of the EOF$_{IE}$ distribution

The mode is obtained by taking the first derivative of the EOF$_{IE}$ density function with respect to *x* i.e. $df(x)/dx=0$. The mode of the EOF$_{IE}$ distribution is given by

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$$\frac{df(x)}{dx}=\left(-\frac{1}{x^4}\alpha\theta\beta e^{-\left(\left(e^{-\frac{\beta}{x}}\right)^{-\alpha}-1\right)^{\theta}}\left(\alpha\beta\left((\theta-1)\left(e^{-\frac{\beta}{x}}\right)^{-\alpha(\theta-1)}-\left(\left(e^{-\frac{\beta}{x}}\right)^{-\alpha}-1\right)^{\theta}\theta\left(e^{-\frac{\beta}{x}}\right)^{-\alpha\theta}\right)\left(1-\left(e^{-\frac{\beta}{x}}\right)^{\alpha}\right)^{\theta-2}\right.\right.$$
$$\left.\left.+2\left(e^{-\frac{\beta}{x}}\right)^{-\alpha\theta}\left(1-\left(e^{-\frac{\beta}{x}}\right)^{\alpha}\right)^{\theta-1}\left(\frac{\alpha\theta\beta}{2}+x\right)\right)\right)=0 \tag{21}$$

It cannot be solved analytically because it does not have a closed form, hence can only be obtained numerically.

5.2 Moment and moment generating function

In this subunit, the ordinary moment (OM), moment generating function (MGF), and incomplete moment of the EOF_{IE} distribution are presented for a random variable (r.v) *X*.

Theorem 1. If $X \sim EOF_{IE}(\xi)$ with $\xi=(\alpha,\theta,\beta)$, then the rth ordinary moment of *X* is expressed as

$$\mu_r'(x)=\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\vartheta_{i,j,k,m,q}\frac{\Gamma(1-r)}{\left(\beta[q+1]\right)^{1-r}}, r<1 \tag{22}$$

Proof:

Let *X* be a r.v following the EOF_{IE} distribution, the rth OM of *X* can be derived as follows:

$$\mu_r'=\mathrm{E}\left(X^r\right)=\int_{-\infty}^{+\infty}x^r f(x,\xi)dx \tag{23}$$

Inserting (15) in (23) gives

$$\mu_r'=\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\vartheta_{i,j,k,m,q}\int_0^{+\infty}x^{r-2}e^{\left[-\beta(l+1)\right]x^{-1}}dx \tag{24}$$

Let $y=\beta(q+1)x^{-1}$, then

$$\mu_r'=\left[\beta(q+1)\right]^{r-1}\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\vartheta_{i,j,k,m,q}\int_0^{+\infty}y^{-r}e^{-y}dy \tag{25}$$

Hence, the OM is given as

$$\mu_r'(x)=\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\vartheta_{i,j,k,m,q}\frac{\Gamma(1-r)}{\left(\beta[q+1]\right)^{1-r}}, r<1 \tag{26}$$

where $\vartheta_{i,j,k,m,q}=\alpha\theta\beta\frac{(-1)^{i+k+m+q}}{i!}\binom{\alpha\theta(i+1)+j}{j}\binom{\theta(i+1)-1}{k}\binom{\alpha k}{m}\binom{m+j}{q}$. The higher order moments can be obtained by substituting $r=1,2,3,\ldots$ in the OMs (22).

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Theorem 2. If $X \sim EOF_{IE}\left(\xi\right)$, then the MGF of *X* is expressed as

$$M_X\left(t\right)=\sum_{r=0}^{\infty}\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\frac{t^r}{r!}\frac{\vartheta_{i,j,k,m,q}\Gamma\left(1-r\right)}{\left(\beta\left[q+1\right]\right)^{1-r}},r<1 \tag{27}$$

Proof:

The MGF is defined as

$$M_X\left(t\right)=\mathrm{E}\left(e^{tX}\right)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{+\infty}x^r f\left(x,\xi\right)dx=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'\left(x\right) \tag{28}$$

Inserting (22) in (28) gives

$$M_X\left(t\right)=\sum_{r=0}^{\infty}\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\frac{t^r}{r!}\frac{\vartheta_{i,j,k,m,q}\Gamma\left(1-r\right)}{\left(\beta\left[q+1\right]\right)^{1-r}},r<1 \tag{29}$$

where $\vartheta_{i,j,k,m,q}=\alpha\theta\beta\frac{\left(-1\right)^{i+k+m+q}}{i!}\binom{\alpha\theta\left(i+1\right)+j}{j}\binom{\theta\left(i+1\right)-1}{k}\binom{\alpha k}{m}\binom{m+j}{q}$.

The conditional moments of the $\mathrm{EOF_{IE}}$ distribution are derived given the importance of the first incomplete moment (IM) $\left(\varphi_1'\right)$. The s[th] lower and upper IMs of X are defined by

$$\varphi_s'\left(t\right)=\mathrm{E}\left(X^s\middle|X<t\right)=\int_0^t x^s f\left(x,\xi\right)dx \tag{30}$$

$$\tau_s\left(t\right)=\mathrm{E}\left(X^s\middle|X>t\right)=\int_t^{+\infty}x^s f\left(x,\xi\right)dx \tag{31}$$

For any $s\in\mathrm{N}$, the s[th] lower IM of $\mathrm{EOF_{IE}}$ distribution is obtained by inserting (15) in (30). We have

$$\varphi_s'\left(t\right)=\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\vartheta_{i,j,k,m,q}\int_0^t x^{s-2}e^{-\beta(q+1)x^{-1}}dx \tag{32}$$

$$\varphi_s'\left(t\right)=\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\vartheta_{i,j,k,m,q}\left[\frac{v\left(1-s,\beta\left(q+1\right)t^{-1}\right)}{\left(\beta\left(q+1\right)\right)^{1-s}}\right],s<1 \tag{33}$$

where, $v\left(s,t\right)=\int_0^t x^{s-1}e^{-x}dx$ is the lower incomplete gamma function. Also, the s[th] upper IM of $\mathrm{EOF_{IE}}$ distribution is obtained by inserting (15) in (31). We have

$$\tau_s'\left(t\right)=\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\vartheta_{i,j,k,m,q}\int_t^{\infty}x^{s-2}e^{-\beta(q+1)x^{-1}}dx \tag{34}$$

$$\tau_s'\left(t\right)=\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\vartheta_{i,j,k,m,q}\left[\frac{\Gamma\left(1-s,\beta\left(q+1\right)t^{-1}\right)}{\left(\beta\left(q+1\right)\right)^{1-s}}\right],s<1 \tag{35}$$

where, $\Gamma\left(s,t\right)=\int_t^{\infty}x^{s-1}e^{-x}dx$ is the upper incomplete gamma function.

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The IMs are used in the calculation of other valuable statistical measures such as the mean deviation about the mean $\delta_1 = \mathrm{E}\left(\left|X-\mu_1'\right|\right)$ and about the median $\delta_2 = \mathrm{E}\left(\left|X-M\right|\right)$. The first IM given as $\left(\varphi_1'\right)$ is useful in defining the mean deviation of X about the mean $\left(\mu_1'\right)$ and median $\left(M\right)$.

$$\delta_1 = \mathrm{E}\left(\left|X-\mu_1'\right|\right) = \int_0^{+\infty}\left|x-\mu_1'\right| f\left(x;\xi\right)dx = 2\mu_1' F\left(\mu_1'\right) - 2\varphi_1'\left(\mu_1'\right), \qquad (36)$$

$$\delta_2 = \mathrm{E}\left(\left|X-\mathrm{M}\right|\right) = \int_0^{+\infty}\left|x-\mathrm{M}\right| f\left(x;\xi\right)dx = \mu_1' - 2\varphi_1'\left(\mathrm{M}\right)$$

where

$\mu_1' = \mu$ is the mean found by injecting $r=1$ in the OM (22).

M is the median gotten by inserting $u=0.5$ in the quantile function (8).

$\varphi_1'\left(t\right) = \int_0^t xf\left(x\right)dx$ is the first IM which can be obtained by inserting $s=1$ in the IM (33).

5.3 Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves (BLCs) are measures for income inequality. There applicability is considered useful to other areas like medicine, reliability, insurance and demography. The BLCs can be expressed in terms of the quantile function of a given distribution. The Bonferroni curve for the $\mathrm{EOF_{IE}}$ distribution using the quantile function (8) is given by

$$\begin{aligned}\mathrm{B}\left(u\right) &= \frac{1}{u\mu}\varphi_1'\left[Q\left(u;\xi\right)\right] \\ &= \frac{1}{u\mu}\varphi_1'\left[-\frac{\beta}{\log\left[\left(1+\left(-\log\left(u\right)\right)^{\frac{1}{\theta}}\right)^{-\frac{1}{\alpha}}\right]}\right], \quad u\in\left(0,1\right)\end{aligned} \qquad (37)$$

and the Lorenz curve is given by

$$\begin{aligned}L\left(u\right) &= \mu\mathrm{B}\left(u\right) = \frac{\mu}{u\mu}\varphi_1'\left[Q\left(u;\xi\right)\right] \\ &= \frac{1}{u}\varphi_1'\left[-\frac{\beta}{\log\left[\left(1+\left(-\log\left(u\right)\right)^{\frac{1}{\theta}}\right)^{-\frac{1}{\alpha}}\right]}\right], \quad u\in\left(0,1\right)\end{aligned} \qquad (38)$$

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5.4 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution and $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ are order statistics gotten from the sample. According to David (1981), the pdf, $f_{p:n}(x)$ of the pth order statistic $X_{p:n}$ is defined as

$$f_{p:n}(x) = \frac{f(x)}{\mathrm{B}(p, n-p+1)} \left[F(x)\right]^{p-1} \left[1 - F(x)\right]^{n-p} \tag{39}$$

where, $G(x)$ and $g(x)$ are the CDF and PDF of the EOF$_{IE}$ distribution respectively, and $\mathrm{B}(.,.)$ is the Beta function. Since $0 < F(x) < 1$ for $x > 0$, expanding $\left[1 - F(x)\right]^{n-p}$ gives

$$f_{p:n}(x) = \frac{1}{\mathrm{B}(p, n-p+1)} \sum_{i=0}^{n-p} (-1)^i \binom{n-p}{i} \left[F(x)\right]^{p+i-1} f(x) \tag{40}$$

Therefore, inserting (6) and (7) in (40) and expanding based on the binomial series expansion, the PDF of the pth order statistic for EOF$_{IE}$ distribution is given as

$$f_{p:n}(x) = \frac{1}{\mathrm{B}(p, n-p+1)} \sum_{w=0}^{k+s} \frac{\vartheta_{i,j,k,q,s,w}}{x^2} e^{-\frac{\beta(w+1)}{x}} \tag{41}$$

where,

$$\vartheta_{i,j,k,q,s,w} = \alpha\theta\beta \sum_{i=0}^{n-p} \sum_{j,k=o}^{\infty} \sum_{q,s=0}^{\infty} \frac{(-1)^{i+j+q+s+w} (p+i)^j}{j!} \binom{n-p}{i} \binom{\alpha\theta(j+1)+k}{k} \binom{\theta(j+1)-1}{q} \binom{\alpha q}{s} \binom{k+s}{w}.$$

The distribution of the minimum and maximum order statistics can be gotten from (41) by setting $p = 1$ and $p = n$, respectively. Also, the rth moment of the pth order statistic for EOF$_{IE}$ distribution is defined as

$$\mathrm{E}\left(X_{p:n}^r\right) = \int_{-\infty}^{+\infty} x^r f_{p:n}(x;\xi)\, dx \tag{42}$$

By inserting (41) in (42), then

$$\mathrm{E}\left(X_{p:n}^r\right) = \frac{1}{\mathrm{B}(p, n-p+1)} \sum_{w=0}^{k+s} \frac{\vartheta_{i,j,k,q,s,w} \Gamma(1-r)}{\left(\beta(w+1)\right)^{1-r}} \tag{43}$$

where,

$$\vartheta_{i,j,k,q,s,w} = \alpha\theta\beta \sum_{i=0}^{n-p} \sum_{j,k=o}^{\infty} \sum_{q,s=0}^{\infty} \frac{(-1)^{i+j+q+s+w} (p+i)^j}{j!} \binom{n-p}{i} \binom{\alpha\theta(j+1)+k}{k} \binom{\theta(j+1)-1}{q} \binom{\alpha q}{s} \binom{k+s}{w}.$$

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5.5 Probability Weighted Moment (PWM)

The PWM of a r.v X is a very useful mathematical quantity proposed by Greenwood *et al.* (1979).

Theorem 3. The PWM of a r.v $X, \tau_{r,s}$ of the EOF$_{\text{IE}}$ distribution is given as

$$\tau_{r,s} = A^* \frac{\Gamma(1-r)}{\left(\beta[s+q+1]\right)^{1-r}}, r<1 \tag{44}$$

where $A^* = \sum_{i,j,k,m=0}^{\infty} \sum_{q=0}^{m+j} \sum_{p=0}^{s} \sum_{l,t,u,s=0}^{\infty} \vartheta_{i,j,k,m,q} \psi_{p,l,t,u,s}$.

Proof:

The PWM of a r.v X is defined as

$$\tau_{r,s} = \mathrm{E}\left[X^r F(x)^s\right] = \int_{-\infty}^{+\infty} x^r f(x)\left(F(x)\right)^s dx \tag{45}$$

Inserting (6) and (7) in (45), then

$$\tau_{r,s} = \sum_{i,j,k,m=0}^{\infty} \sum_{q=0}^{m+j} \sum_{p=0}^{s} \sum_{l,t,u,s=0}^{\infty} \vartheta_{i,j,k,m,q} \psi_{p,l,t,u,s} \int_0^{+\infty} x^{r-2} e^{\left[-\beta(s+q+1)\right]x^{-1}} dx \tag{46}$$

where $\vartheta_{i,j,k,m,q} = \alpha\theta\beta \frac{(-1)^{i+k+m+q}}{i!} \binom{\alpha\theta(i+1)+j}{j} \binom{\theta(i+1)-1}{k} \binom{\alpha k}{m} \binom{m+j}{q}$,

$\psi_{p,l,t,u,s} = \frac{(-1)^{l+u} (p)^l}{i!} \binom{s}{p} \binom{\theta l+t-1}{t} \binom{\theta l+t}{u} \binom{\alpha u}{s}$.

Let $y = \beta(s+q+1)x^{-1}$, then

$$\tau_{r,s} = \left(\frac{\beta(s+q+1)}{x}\right)^{r-1} A^* \int_0^{+\infty} y^{-r} e^{-y} dy \tag{47}$$

where $A^* = \sum_{i,j,k,m=0}^{\infty} \sum_{q=0}^{m+j} \sum_{p=0}^{s} \sum_{l,t,u,s=0}^{\infty} \vartheta_{i,j,k,m,q} \psi_{p,l,t,u,s}$

The PWM of the EOF$_{\text{IE}}$ in gamma form is given as

$$\tau_{r,s} = A^* \frac{\Gamma(1-r)}{\left(\beta[s+q+1]\right)^{1-r}}, r<1 \tag{48}$$

where $A^* = \sum_{i,j,k,m=0}^{\infty} \sum_{q=0}^{m+j} \sum_{p=0}^{s} \sum_{l,t,u,s=0}^{\infty} \vartheta_{i,j,k,m,q} \psi_{p,l,t,u,s}$.

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5.6 Entropy

The entropy of a r.v X is a measure of the dissimilarity of uncertainty. In this subunit, two widely known entropy measures called the Shannon and Rényi entropies (Shannon, 1984; Rényi, 1961) are considered. The Rényi entropy (REnt) of a r.v X is defined as

$$\mathrm{I}_{R(\delta)} = \frac{1}{1-\delta}\log\int_{-\infty}^{+\infty} f(x)^{\delta}\,dx, \quad \text{where } \delta > 0 \text{ and } \delta \neq 1. \tag{50}$$

Using the similar notion for escalating the EOF$_{\mathrm{IE}}$ density function (PDF), we have

$$f(x)^{\delta} = (\alpha\theta)^{\delta}\sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j} w_{i,j,k,m,q}\left(\frac{\beta}{x^2}e^{-\beta/x}\right)^{\delta}\left(e^{-\beta/x}\right)^{q} \tag{51}$$

where $w_{i,j,k,m,q} = \dfrac{(-1)^{i+k+m+q}\,\delta^{i}}{i!}\binom{\alpha\theta(i+\delta)+\delta+j-1}{j}\binom{\theta(i+\delta)-\delta}{k}\binom{\alpha k}{m}\binom{m+j}{q}$.

Hence, the REnt of X with EOF$_{\mathrm{IE}}$ distribution can be expressed as

$$\mathrm{I}_{R(\delta)} = \frac{1}{1-\delta}\log\left[\eta^{*}\int_{0}^{+\infty} x^{-2\delta}e^{[-\beta(q+\delta)]x^{-1}}\,dx\right] \tag{52}$$

where $\eta^{*} = \sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\dfrac{(\alpha\theta\beta)^{\delta}(-1)^{i+k+m+q}\,\delta^{i}}{i!}\binom{\alpha\theta(i+\delta)+\delta+j-1}{j}\binom{\theta(i+\delta)-\delta}{k}\binom{\alpha k}{m}\binom{m+j}{q}$

Then

$$\eta^{*}\int_{0}^{+\infty} x^{-2\delta}e^{[-\beta(q+\delta)]x^{-1}}\,dx = \eta^{*}\frac{\Gamma(1-2\delta)}{[\beta(q+\delta)]^{2\delta-1}} \tag{53}$$

Thus, the REnt of X with EOF$_{\mathrm{IE}}$ distribution is given as

$$\mathrm{I}_{R(\delta)} = \frac{1}{1-\delta}\log\left[\eta^{*}\frac{\Gamma(1-2\delta)}{[\beta(q+\delta)]^{2\delta-1}}\right] \tag{54}$$

where $\eta^{*} = \sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\dfrac{(\alpha\theta\beta)^{\delta}(-1)^{i+k+m+q}\,\delta^{i}}{i!}\binom{\alpha\theta(i+\delta)+\delta+j-1}{j}\binom{\theta(i+\delta)-\delta}{k}\binom{\alpha k}{m}\binom{m+j}{q}$.

The Shannon entropy (ShEnt) of a r.v X is defined as $S_E = \mathrm{E}[-\log(f(x))]$. The ShEnt is a special case of the REnt when $\delta \uparrow 1$. The $\delta - entropy$ is defined as

$$H_{(\delta)} = \frac{1}{\delta-1}\log\left[1-\int_{-\infty}^{+\infty} f(x)^{\delta}\,dx\right] \tag{55}$$

The ShEnt of X with EOF$_{\mathrm{IE}}$ distribution is given as

$$\mathrm{H}(\delta) = \frac{1}{\delta-1}\log\left[1-\left(\eta^{*}\frac{\Gamma(1-2\delta)}{[\beta(q+\delta)]^{2\delta-1}}\right)\right] \tag{56}$$

where $\eta^{*} = \sum_{i,j,k,m=0}^{\infty}\sum_{q=0}^{m+j}\dfrac{(\alpha\theta\beta)^{\delta}(-1)^{i+k+m+q}\,\delta^{i}}{i!}\binom{\alpha\theta(i+\delta)+\delta+j-1}{j}\binom{\theta(i+\delta)-\delta}{k}\binom{\alpha k}{m}\binom{m+j}{q}$.

5.7 Stress-Strength Reliability

Let X_1 be a system's strength exposed to stress X_2. If X_1 follow the $EOF_{IE}(\alpha_1,\theta_1,\beta_1)$ and X_2 follow the $EOF_{IE}(\alpha_2,\theta_2,\beta_2)$, provided that X_1 and X_2 are statistically independent r.vs, then the stress strength reliability measure which measures the performance of the system is given by

$$R=P\left(X_2<X_1\right)=\int_0^\infty f_1\left(x\right)F_2\left(x\right)dx, \tag{57}$$

$$R=\int_0^\infty \frac{\alpha_1\theta_1\left(\frac{\beta_1}{x^2}e^{-\frac{\beta_1}{x}}\right)\left[1-\left(e^{-\frac{\beta_1}{x}}\right)^{\alpha_1}\right]^{\theta_1-1}}{\left(e^{-\frac{\beta_1}{x}}\right)^{\alpha_1\theta_1+1}} e^{-\left(\left(e^{-\frac{\beta_1}{x}}\right)^{-\alpha_1}-1\right)^{\theta_1}}\times e^{-\left(\left(e^{-\frac{\beta_2}{x}}\right)^{-\alpha_2}-1\right)^{\theta_2}}dx \tag{58}$$

Applying BSE with some mathematical manipulations, (58) can be written as

$$R=\sum_{i,j,k,m,l=0}^{\infty}\sum_{q=0}^{m+j}A^*\int_0^\infty x^{-2}e^{-\left(\beta_2 l+\left(q+1\right)\beta_1\right)x^{-1}}dx \tag{59}$$

where $A^*=\alpha_1\theta_1\beta_1\frac{(-1)^{i+k+m+q}}{i!}\binom{\alpha_1\theta_1\left(i+1\right)+j}{j}\binom{\theta_1\left(i+1\right)-1}{k}\binom{\alpha_1 k}{m}\binom{m+j}{q}\binom{\theta_2 i+j-1}{j}\binom{\theta_2 i+j}{k}\binom{\alpha_2 k}{l}$

Thus, the stress strength reliability of EOF_{IE} is given as

$$R=\sum_{i,j,k,m,l=0}^{\infty}\sum_{q=0}^{m+j}\frac{A^*}{\left[\beta_2 l+\left(q+1\right)\beta_1\right]} \tag{60}$$

6. Parameter Estimation

The ML estimates for the unknown parameters of the EOF_{IE} distribution are obtained from the complete samples as follows. Suppose a random sample has $X_1,X_2,\ldots,X_n$ as possible outcomes acquired from the EOF_{IE} distribution with $\xi=\left(\alpha,\theta,\beta\right)^T$ as an unknown parameter vector. Let l be the log-likelihood function of the EOF_{IE} then

$$\begin{aligned} l=\log L\left(\xi\right)&=n\log\alpha+n\log\theta+n\log\beta-2\sum_{i=1}^{n}\log x_i+\left(\theta+1\right)\sum_{i=1}^{n}\log\left(1-\left(e^{-\beta/x_i}\right)^{\alpha}\right)\\ &+\beta\left(\alpha\theta+1\right)\sum_{i=1}^{\infty}\frac{1}{x_i}-\sum_{i=1}^{n}\left(\left(e^{-\beta/x_i}\right)^{-\alpha}-1\right)^{\theta}\end{aligned} \tag{61}$$

The concomitant score function is set as $U\left(\xi\right)=\left[\frac{\partial l}{\partial\alpha},\frac{\partial l}{\partial\theta},\frac{\partial l}{\partial\beta}\right]^T$. The l can be maximized by elucidating the system nonlinear likelihood equations acquired by differentiating (61). Parts of the score function are

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$$\frac{\partial l}{\partial \alpha}=\frac{n}{\alpha}-(\theta-1)\sum_{i=1}^{n}\frac{\left(e^{-\beta/x_i}\right)^{\alpha}\log\left(e^{-\beta/x_i}\right)}{1-\left(e^{-\beta/x_i}\right)^{\alpha}}+\theta\beta\sum_{i=1}^{n}\frac{1}{x_i}+\theta\sum_{i=1}^{n}\frac{\left(\left(e^{-\beta/x_i}\right)^{-\alpha}-1\right)^{\theta-1}\log\left(e^{-\beta/x_i}\right)}{\left(e^{-\beta/x_i}\right)^{\alpha}}$$

$$\frac{\partial l}{\partial \theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log\left(1-\left(e^{-\beta/x_i}\right)^{\alpha}\right)+\alpha\beta\sum_{i=1}^{n}\frac{1}{x_i}-\sum_{i=1}^{n}\left(\left(e^{-\beta/x_i}\right)^{-\alpha}-1\right)^{\theta}\log\left(\left(e^{-\beta/x_i}\right)^{\alpha}-1\right)$$

and

$$\frac{\partial l}{\partial \beta}=\frac{n}{\beta}+\beta(\theta-1)\sum_{i=1}^{n}\frac{\frac{1}{x_i}\left(e^{-\beta/x_i}\right)^{\alpha}}{1-\left(e^{-\beta/x_i}\right)^{\alpha}}+(\alpha\theta+1)\sum_{i=1}^{n}\frac{1}{x_i}-\alpha\theta\sum_{i=1}^{n}\frac{\frac{1}{x_i}\left(\left(e^{-\beta/x_i}\right)^{-\alpha}-1\right)^{\theta-1}}{\left(e^{-\beta/x_i}\right)^{\alpha}}$$

The ML estimate of $\alpha, \theta,$ and β cannot be solved analytically because the nonlinear system of equations exists in an unclosed form, so in this research R-statistical software (*Newdistns* and *AdequacyModel* packages) is used to solve them numerically via iterative methods.

7. Simulation Study

Here, the performance of the ML estimates for the EOF_{IE} model is evaluated using simulation study. Performance of the estimators is evaluated through the average estimates (MEs), Absolute biases, mean square errors (MSE) and root mean square errors (RMSE) for different sample sizes. 10,000 samples from the EOF_{IE} distribution are generated, each of sample size n = 25, 75, 150, 250, 350, and 500, for selected values of $\alpha, \theta,$ and β. The absolute Bias, MSE and RMSE are computed for $\xi=\hat{\alpha},\hat{\theta},\hat{\beta}$ using

$$Abs\hat{B}ias_s=\frac{1}{10000}\sum_{i=1}^{10000}\left|\hat{S}_i-S\right|$$

$$\hat{M}SE_s=\frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{S}_i-S\right)^2$$

$$\hat{R}MSE_s=\sqrt{\frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{S}_i-S\right)^2}$$

The simulation results are displayed in Table 2. The results shows that the MSE and RMSE for the estimated parameter values decreases as n increases. Therefore, as n increases, the average estimates tend towards the true values of the parameters.

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Table 2: Simulation study for EOF_{IE} model

n	Average estimates			Absolute Bias			MSE			RMSE		
	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\alpha}$
$\theta = 0.5, \beta = 0.6, \alpha = 0.9$												
25	0.5301	0.689	0.9440	0.0301	0.0089	0.0440	0.0125	0.0192	0.0072	0.1118	0.1384	0.0850
75	0.5097	0.5873	0.9395	0.0097	0.0127	0.0395	0.0035	0.0062	0.0031	0.0588	0.0789	0.0554
150	0.5043	0.5828	0.9374	0.0043	0.0172	0.0374	0.0016	0.0033	0.0022	0.0401	0.0573	0.0472
250	0.5025	0.5801	0.9369	0.0025	0.0199	0.0369	0.0009	0.0021	0.0019	0.0305	0.0462	0.0441
350	0.5019	0.5799	0.9355	0.0019	0.0201	0.0355	0.0007	0.0016	0.0018	0.0259	0.0405	0.0421
500	0.5013	0.5797	0.9347	0.0013	0.0203	0.0347	0.0005	0.0013	0.0016	0.0216	0.0355	0.0405
$\theta = 1.0, \beta = 1.2, \alpha = 1.5$												
25	1.0620	1.1910	1.5510	0.0615	0.0088	0.0512	0.0420	0.0267	0.0136	0.2049	0.1634	0.1168
75	1.0200	1.1670	1.5580	0.0198	0.0332	0.0575	0.0114	0.0101	0.0068	0.1070	0.1004	0.0825
150	1.0090	1.1620	1.5570	0.0090	0.0377	0.0570	0.0053	0.0057	0.0053	0.0730	0.0757	0.0729
250	1.0050	1.1620	1.5530	0.0052	0.0377	0.0533	0.0031	0.0040	0.0045	0.0556	0.0629	0.0668
350	1.0040	1.1630	1.5520	0.0039	0.0373	0.0516	0.0022	0.0032	0.0041	0.0471	0.0566	0.0642
500	1.0030	1.1640	1.5480	0.0027	0.0356	0.0484	0.0015	0.0025	0.0037	0.0391	0.0504	0.0607
$\theta = 1.5, \beta = 2.0, \alpha = 2.5$												
25	1.5920	1.9670	2.583	0.0919	0.0334	0.0829	0.0890	0.0436	0.0269	0.2983	0.2087	0.1640
75	1.5300	1.9440	2.5860	0.0297	0.0557	0.0859	0.0240	0.0149	0.0126	0.1549	0.1222	0.1112
150	1.5140	1.9440	2.5810	0.0136	0.0565	0.0810	0.0112	0.0094	0.0101	0.1057	0.0967	0.1003
250	1.5080	1.9470	2.5730	0.0078	0.0528	0.0726	0.0065	0.0068	0.0080	0.0805	0.0825	0.0895
350	1.5060	1.9500	2.5680	0.0059	0.0504	0.0680	0.0046	0.0054	0.0070	0.0682	0.0737	0.0835
500	1.5040	1.9520	2.5630	0.0040	0.0475	0.0734	0.0032	0.0044	0.0061	0.0566	0.0662	0.0778

8. Application

A demonstration of the EOF_{IE} distribution flexibility using two real datasets and compared the new model with some existing models in the literature is presented in this section. The distribution's parameters estimated by ML estimation technique, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Consistent Akaike Information Criterion (CAIC), Hannan-Quinn information Criterion (HQIC), Anderson Darling statistic (A*), Cramer-von Mises statistic (W*), and Kolmogorov-Smirnov test (KS) are calculated to compare the fitted distributions based on each dataset. The fitted distribution with the smallest values in terms of the AIC, BIC, CAIC and HQIC fits the data better than the others.

The first data represents 47 years North Saskachevan highest yearly flood discharges (1000 cf/sec) at the North Saskachevan River in Edmonton. This dataset has been analysed by Montfort (1970) and Hag and Elgarhy (2018).

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19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560.

The flexibility of the new EOF_{IE} model in contrast with some existing distributions, which includes the Gompertz Lomax (GOLO), Odd Fréchet Inverse exponential (OFIE), Kumaraswamy Inverse exponential (KUIE), Inverse exponential (IE), generalized inverse exponential (GIE), exponential inverse exponential (EIE), is exhibited with the first dataset. The descriptive statistics of the first dataset are provided in Table 3. From the tables, we observed that the dataset is positively skewed.

Table 3: Descriptive statistics for the highest yearly flood discharges dataset.

Mean	**Median**	**Min**	**Max**	**1^{st} Qu**	**3^{rd} Qu**	**skewness**	**Kurtosis**
51.5	40.4	19.9	185.6	30.3	61.3	2.004	4.623

Table 4 and 5 presents the models estimated values and the goodness of fit measures like AIC, CAIC, BIC, HQIC, A* and W* statistics while the KS statistics with its *p*-values are presented in Table 6, for comparison of the fitted distributions. The ranks in Table 5 are based on the AIC, CAIC, BIC, HQIC goodness-of-fit measure from the lowest to the highest values.

Table 4: Estimated values for the highest yearly flood discharges (First dataset).

Model	**Estimates**			
	α	β	θ	γ
$EOF_{IE}(\alpha,\beta,\theta)$	4.878	4.878	1.802	-
$EIE(\alpha,\beta)$	6.31	6.31	-	-
$GIE(\alpha,\beta)$	5.577	93.983	-	-
$KUIE(\alpha,\beta,\theta)$	9.695	5.579	9.695	-
$OFIE(\alpha,\beta)$	28.717	0.819	-	-
$GOLO(\alpha,\beta,\theta,\gamma)$	0.580	0.446	0.007	3.996
$IE(\beta)$	-	39.81	-	-

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Table 5: Goodness-of-fit statistics for the highest yearly flood discharges (First dataset).

Model	AIC	BIC	CAIC	HQIC	A*	W*	-log (Likelihood)	Rank
$EOF_{IE}(\alpha,\beta,\theta)$	436.8	442.4	437.3	438.9	0.196	0.029	215.4	1
$EIE(\alpha,\beta)$	475.6	479.4	475.9	477.0	0.253	0.036	235.8	7
$GIE(\alpha,\beta)$	437.5	443.3	437.8	439.0	0.385	0.056	216.8	2
$KUIE(\alpha,\beta,\theta)$	439.5	445.2	440.1	441.7	0.385	0.056	216.8	3
$OFIE(\alpha,\beta)$	451.7	455.4	451.9	453.1	0.129	0.018	223.8	4
$GOLO(\alpha,\beta,\theta,\gamma)$	460.5	468.0	461.4	463.3	1.771	0.284	226.2	5
$IE(\beta)$	473.6	475.5	473.7	474.3	0.253	0.036	235.8	6

Table 6: KS statistics for the highest yearly flood discharges (First dataset).

Model	KS	*p*-value
$EOF_{IE}(\alpha,\beta,\theta)$	0.072	1.0
$EIE(\alpha,\beta)$	0.29	8e-04
$GIE(\alpha,\beta)$	0.085	0.9
$KUIE(\alpha,\beta,\theta)$	0.085	0.9
$OFIE(\alpha,\beta)$	0.19	0.06
$GOLO(\alpha,\beta,\theta,\gamma)$	0.15	0.2
$IE(\beta)$	0.29	8e-04

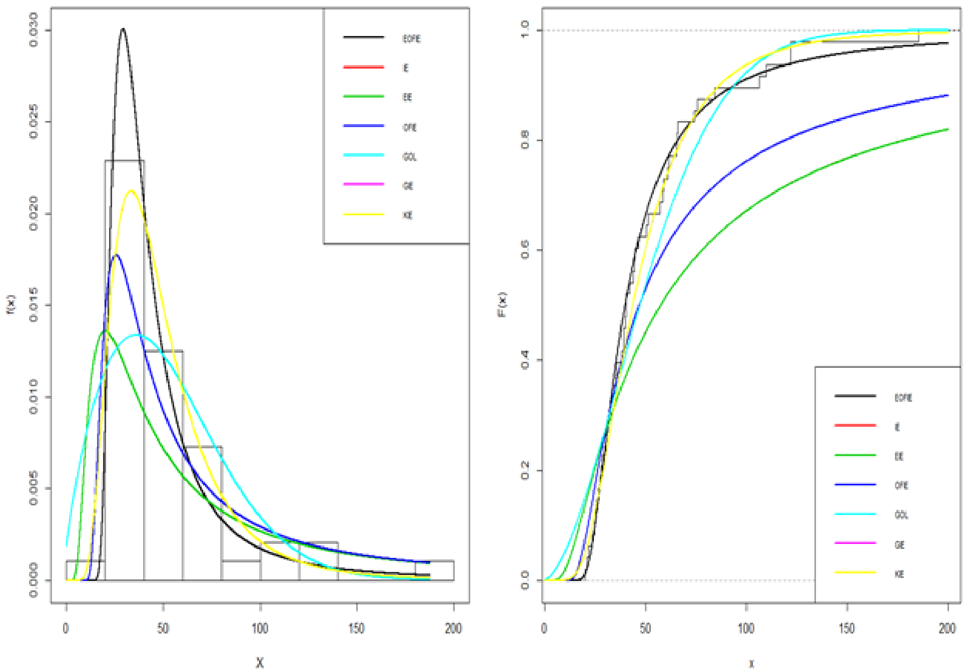

Figure 3: Plots of the histogram and estimated density (Left), empirical and estimated cdfs (Right) of EOF_{IE} distribution for the first dataset (highest yearly flood discharges).

The second data signifies the strength measured in GPA for single carbon fibres (SCF) and impregnated 1000-carbon fibres tows. The SCF were tested under tension at gauge lengths of 10 mm with 63 observations. The data has been used by Badar and Priest (1982), Afify *et al.* (2015a), Afify *et al.* (2015b) and Mead *et al.* (2017).

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

The second data is used to compare the flexibility of the new EOF_{IE} model with some existing models, which includes the Gompertz Lomax (GOLO), Odd Fréchet Inverse exponential (OFIE), Gompertz Inverse exponential (GOIE), Inverse exponential (IE), Gompertz exponential (GOE), exponential inverse exponential (EIE) distributions. The descriptive statistics of the second dataset are provided in Table 7.

Table 7: Descriptive statistics for SCF and impregnated 1000-carbon fibres tows dataset.

Mean	Median	Min	Max	1st Qu	3rd Qu	Skewness	Kurtosis
3.06	3.00	1.90	5.02	2.55	3.42	0.618	0.183

Table 8 and 9 presents the models estimated values and the goodness of fit measures like AIC, CAIC, BIC, HQIC, A* and W* statistics while the KS statistics with its *p*-values are presented in Table 10, for comparison of the fitted models. The ranks in Table 9 are based on the AIC, CAIC, BIC, HQIC goodness-of-fit measure from the lowest to the highest values.

Table 8: Estimated values for the SCF and impregnated 1000-carbon fibres tows (Second dataset).

Model	Estimates			
	α	β	θ	γ
$EOF_{IE}(\alpha,\beta,\theta)$	1.369	1.369	3.903	-
$EIE(\alpha,\beta)$	1.715	1.715	-	-
$OFIE(\alpha,\beta)$	0.941	2.320	-	-
$GOLO(\alpha,\beta,\theta,\gamma)$	0.453	1.455	0.004	5.316
$IE(\beta)$	2.942	-	-	-
$GOIE(\alpha,\beta,\theta)$	0.008	1.339	6.045	-
$GOE(\alpha,\beta,\theta)$	0.016	0.788	1.771	-

https://doi.org/10.22452/josma.vol5no1.1

Table 9: Goodness-of-fit statistics for the SCF and impregnated 1000-carbon fibres tows (Second dataset).

Model	AIC	BIC	CAIC	HQIC	A*	W*	-log (Likelihood)	Rank
$EOF_{IE}(\alpha,\beta,\theta)$	126.2	132.7	126.6	128.8	0.769	0.134	60.11	1
$EIE(\alpha,\beta)$	270.8	275.1	271.0	272.5	0.322	0.060	133.4	7
$GOIE(\alpha,\beta,\theta)$	132.8	139.2	133.2	135.3	1.025	0.149	63.4	2
$GOE(\alpha,\beta,\theta)$	144.3	150.7	144.7	146.8	1.693	0.258	69.15	4
$OFIE(\alpha,\beta)$	229.8	234.1	230.0	231.5	0.378	0.073	112.9	5
$GOLO(\alpha,\beta,\theta,\gamma)$	137.9	146.5	138.6	141.3	1.113	0.162	64.96	3
$IE(\beta)$	268.8	271.0	268.9	269.7	0.322	0.060	133.4	6

Table 10: KS statistic for the SCF and impregnated 1000-carbon fibres tows (Second dataset).

Model	KS	***p*-value**
$EOF_{IE}(\alpha,\beta,\theta)$	0.1	0.5
$EIE(\alpha,\beta)$	0.47	1e-12
$GOIE(\alpha,\beta,\theta)$	0.1	0.5
$GOE(\alpha,\beta,\theta)$	0.14	0.2
$OFIE(\alpha,\beta)$	0.47	1e-12
$GOLO(\alpha,\beta,\theta,\gamma)$	0.13	0.3
$IE(\beta)$	0.47	1e-12

It is observed that the EOF_{IE} model gives the lowest goodness of fit measures in the two datasets which imply that the EOF_{IE} distribution performs superior than the other competing distributions. Furthermore, the (i) histogram and estimated PDF, (ii) empirical and estimated CDFs of the EOF_{IE} distribution of the first and second datasets which are displayed in Figures 3 and 4, illustrate the flexibility and superiority of the EOF_{IE} distribution over the other present distributions in this article.

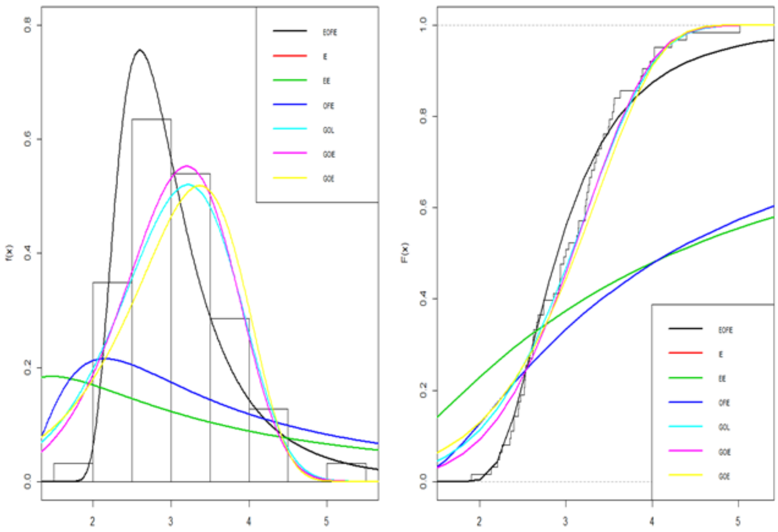

Figure 4: Plots of the histogram and estimated density (Left), empirical and estimated cdfs (Right) of EOF_{IE} distribution for the second dataset (SCF and impregnated 1000-carbon fibres tows).

9. Conclusion

This paper introduced a novel three parameter model called the extended odd Fréchet inverted exponential (EOF_{IE}) distribution. The overt mathematical lingos of some of the important properties of the new model such as the ordinary and incomplete moments, probability weighted moments, quantile function, Rényi and Shannon entropies, order statistics, Bonferroni and Lorenz curves are investigated. The distribution parameters are estimated by the ML technique and the finite sample performance assessed through simulation. Finally, two applications using two real-life datasets illustrate that the EOF_{IE} distribution delivers a superior fit than the other existing viable models. In conclusion, the EOF_{IE} distribution offers a very flexible model for analysing positive datasets arising in environmental and engineering fields as well as several other areas of scientific inquiries.

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