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Bayesian Restricted Stein-Rule Least Squares with Non-Spherical Disturbances

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RESEARCH ARTICLE

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Abstract

Bayesian restricted Stein-rule least squares is a novelty estimator introduced to obtain parameters of the family of restricted least squares with intertwined heteroscedastic and autocorrelated disturbance errors. Errors in regression modelling set to measure the quality of the data and/or model of which intertwining of most prominent errors disturbances play a significant role in determining the quality of the data and/or model. Efforts were geared towards the comparison of the performances and relative efficiency of the small sample property of Bayesian and classical family of restricted Stein-rule least squares. The sample size was set at 25 to capture the intertwined disturbance errors, and the iteration of the Monte Carlo simulation was set at 10000 for both classical and Bayesian paradigms. The ρ and $\emptyset$ for both Autoregressive of order one (AR(1)) and Moving average of order one (MA(1)) processes respectively were set asymmetrically as -0.8,-0.5,-0.3, 0,0.3,0.5 and 0.8 while δ as a scale of heteroscedasticity was set at 0-homoscedasticity, -0.3-mild,-0.5-moderate and 1-severe. The outcome of the study pointed out that Bayesian estimation (posterior mean and Bayes estimates) for restricted least squares and restricted Stein-rule estimators outperformed classical restricted and restricted Stein-rule estimators both in performances and relative efficiency measurements. It is therefore recommended to make use of the Bayesian framework when encountering similar disturbances in a small sample size.

Keywords: Autocorrelated error, Bayesian and intertwine JEL Classification: C01, C11 and C15, Heteroscedasticity, Restricted, Stein-rule

1. Introduction

Errors in regression modelling set to measure the quality of the data and/or model of which intertwining of most prominent error disturbances play a significant role in determining the quality and standard of the data and/or model. In econometrics research, the prominent disturbances usually faced by the analyst are autocorrelated and heteroscedastic disturbance errors. Independently, studies had been carried out on Bayesian estimation with heteroscedasticity disturbance error, Oloyede et al. (2013), Tanizaki (2003), Chib (1993), Chib and Greenberg (1994) where it was affirmed that the presence of such disturbance error in the data or model renders inferences of the parameter estimates invalid.

Adjibolosoo (1993) investigated the efficiency of estimators under various heteroscedastic error structures and concluded that an exponential family of heteroscedastic error structures would bring about efficient parameter estimates for improved statistical inference. Both heteroscedasticity and autocorrelated disturbance errors violated the assumption of ordinary least squares and render it inefficient and inconsistent. Balu and Harry (1983) examined a Monte Carlo study of linear regression with intertwined autocorrelated and heteroscedastic disturbance errors, thus found out that when the magnitude of sample size is small, ordinary least squares are as good as other estimators when measured with mean square error. But if the magnitude of the sample is large their modified Durbin and U-hat estimators outperformed all other estimators considered in their studies. This idea contravenes the idea of Bayesian inference which pointed out that a large sample size makes both classical and Bayesian converge.

Liya and You-Gan (2021) examined regression with asymmetric loss function and found out that their proposed method outperformed other algorithms considered in their study. Özbay et al. (2017) examined both multicollinearity and autocorrelation of error as twin non-spherical disturbances simultaneously, their study concluded that their proposed estimator outperformed other estimators deemed in a Monte Carlo simulation paradigm for both AR(1) and MA(1) processes. Chaturvedi et al. (1993) examined the performance of the Stein-rule estimator with mis-specified spherical disturbances and observed the effect of non-spherical disturbance in the dominant condition underlying Stein-rule estimators. Their study derived asymptotic risk properties of the estimator under a quadratic loss function.

Mary et al. (2011) examined generalized partial linear models with shape-restricted splines and pointed out that their method has desirable Bayesian and frequentist properties. Their method performed similar to standard parametric methods when the parametric assumptions are met and is superior when the assumptions are violated. Scott and Valen (2016) examined the restricted powerful Bayesian test and found out that the rejection region is more or less similar to the frequentist F-test. Oloyede (2022) compared the Bayesian and classical families of restricted least squares estimators when covariates are collinear and found out that Bayesian estimates (posterior mean and Bayes estimates) outperformed classical restricted least squares. Lewis et al. (2021) examined Bayesian restricted likelihood methods having outliers' datasets and model misspecification, thus provided an alternative approach to the drawback identified from the previous studies they reviewed, they are of the opinion that their proposed estimator has the better predictive solution.

The contribution of this study is on the premise that the small sample property of restricted Stein-rule estimators with an intertwining of heteroscedasticity and autocorrelated error has not been featured in the literature for both classical and Bayesian framework. This study fills this gap. The paper is arranged as follows: sequel to section one which covered the introductory aspect of the study is the section 2 which discussed model design that captures autocorrelated error of both AR(1) and MA(1) processes, then we have heteroscedasticity, model performance metrics and design of Monte Carlo simulation. Section 3 discussed data analysis and interpretation while the section four discusses the conclusion.

2. Restricted Stein-rule Design

Let $y = X\beta + u$ be the linear regression model where y is an $n \times 1$ set of observations on the regressand, X is a set of $n \times p$ full column rank of regressors, β is $p \times 1$ vectors of unknown parameters while u is an $n \times 1$ vectors of disturbance error not necessarily well behave which ordinarily intertwine with both heteroscedastic and autocorrelated errors.

Let there be *m* linearly independent restriction that constrain the regression coefficients such that

$$r = R\beta$$

where r is an $m \times 1$ vector and R is an $m \times p$ matrix of rank $m < p$. The AR(1) with heteroscedastic is expressed as

$$\widehat{\Omega}\widehat{\Psi}_{AR} = \frac{\sigma_u^2}{1-\rho^2}\begin{bmatrix} \Omega_{11} & \rho & \rho^2 & . & . & . & \rho^{n-1} \\ \rho & \Omega_{12} & \rho & . & . & . & \rho^{n-2} \\ \rho^2 & \rho & \Omega_{13} & . & . & . & \rho^{n-3} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & & . & . & . \ \Omega_{1n} \end{bmatrix}$$

The MA(1) with heteroscedastic is expressed as

$$\widehat{\Omega}\widehat{\Psi}_{MA} = \sigma_u^2\begin{bmatrix} (1+\phi^2)\Omega_{11} & \phi & . & . & . & . & . \\ \phi & (1+\phi^2)\Omega_{12} & \phi & . & . & . & . \\ . & \phi & (1+\phi^2)\Omega_{13} & \phi & . & . & . \\ & . & . & . & . & . & . \\ & . & . & . & . & . & . \\ & . & . & . & . & . & . \\ & . & . & . & . & . & (1+\phi^2)\Omega_{1n} \end{bmatrix}$$

ϕ ranges between -1 and 1, the generalized restricted least squares (GRLS) estimates is obtained as follow, adopting the criterion of minimising the sum of squares $(y - X\beta)'\widehat{\Omega}\widehat{\Psi}(y - X\beta)$ subject to the condition that $R\beta = r$. This leads to the Lagrangean function

$$\begin{aligned} L &= (y - X\beta)'\widehat{\Omega}\widehat{\Psi}(y - X\beta) + 2\lambda'(R\beta - r) \\ &= y'\widehat{\Omega}\widehat{\Psi}y - 2y'\widehat{\Omega}\widehat{\Psi}X\beta + \beta'X'\widehat{\Omega}\widehat{\Psi}X\beta + 2\lambda'R\beta - 2'r \end{aligned} \tag{2.1}$$

Differentiate L with respect to β and setting it to zero

$$-2y'\widehat{\Omega}\widehat{\Psi}X + 2\beta'X'\widehat{\Omega}\widehat{\Psi}X + 2\lambda'R = 0$$

$$-y'\widehat{\Omega}\widehat{\Psi}X + \beta'X'\widehat{\Omega}\widehat{\Psi}X + \lambda'R = 0$$

$$\beta'X'\widehat{\Omega}\widehat{\Psi}X = X'\widehat{\Omega}\widehat{\Psi}y - \lambda'R$$

$$\beta = \left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}X'\widehat{\Omega}\widehat{\Psi}y - \left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}\lambda'R \tag{2.2}$$

$$\hat{\beta}_R = \hat{\beta} - \left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}\lambda'R \tag{2.3}$$

where $\hat{\beta}$ is unrestricted least squares estimates, note that $r = R\hat{\beta}_R$, multiply both sides by R we have

$$\widehat{R\beta}_R = R\hat{\beta} - R\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}\lambda'R \tag{2.4}$$

$$r = R\hat{\beta} - R\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}\lambda'R$$

$$r - R\hat{\beta} = -R\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}\lambda'R$$

$$\left[R\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}R\right]^{-1}\left(R\hat{\beta} - r\right) = \lambda$$

$$\hat{\beta}_R = \hat{\beta} - \left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}\left[R\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}R\right]^{-1}R\left(R\hat{\beta} - r\right)$$

$$\hat{\beta}_R = \hat{\beta} + \left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}\left[R\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}R\right]^{-1}R\left(r - R\hat{\beta}\right) \quad (2.5)$$

thus $\hat{\beta}_R$ is a constrained estimate, following Chaturvedi et al. (2001), the restricted Stein-rule version of interwine disturbance errors can be expressed as:

$$\hat{\beta}_S = \left[1 - \frac{a}{n}\frac{(y-X\hat{\beta})'\widehat{\Omega}\widehat{\Psi}(y-X\hat{\beta})}{\hat{\beta}X\widehat{\Omega}\widehat{\Psi}X\hat{\beta}}\right]\hat{\beta}$$

From the above Stein-rule estimator, restricted Stein-rule can be expressed as:

$$\hat{\beta}_{RS} = \hat{\beta}_S + \left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}R'\left[R\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}R'\right]^{-1}\left(r - R\hat{\beta}_S\right) \quad (2.6)$$

where $\widehat{\Omega}\widehat{\Psi} = \Omega\Psi(b)$, b is a consistent and efficient estimator of β, thus $b = \left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}\left(X'\widehat{\Omega}\widehat{\Psi}y\right)$, following Chaturvedi and Shukla (1990), we have modified Stein-rule estimator of β.

2.1 Bayesian Restricted Least Squares Estimator

Theorem: the restricted posterior density of restricted β and σ is obtained through marginalising of the conjugate of normal-inverse gamma and restricted likelihood since both are of the same family of distribution Oloyede (2022).

The linear model is express as:

$$y = X\beta_R + u \quad (2.7)$$

The likelihood function of β ,X and y , where $\theta = (\beta, \lambda)$ given sample vectors $X_1, X_2 = (1,2,\dots,n)'$ and $y = (y_1, y_2, \dots, y_n)'$ and incorporating restricted β_R is expressed as

$$L(\beta_R, \sigma^2|X, y) = (2\pi\sigma^2)^{-\frac{n}{2}}exp\left[-\frac{1}{2\sigma^2}(y - X\beta_R)'\widehat{\Omega}\widehat{\Psi}(y - X\beta_R)\right] \quad (2.8)$$

Simplify further, $(2\pi)^{-\frac{n}{2}}$ is omitted since it is a constant which ordinary contribute no significant effect, Oloyede (2022)

$$L(\beta_R, \sigma^2|X, y) \propto \sigma^{-n}exp\left[-\frac{1}{2\sigma^2}\left(y'\widehat{\Omega}\widehat{\Psi}y - 2\beta_R{}'X'\widehat{\Omega}\widehat{\Psi}y + \beta_R{}'X'\widehat{\Omega}\widehat{\Psi}X\beta_R\right)\right] \quad (2.9)$$

$$L(\beta_R, \sigma^2|X, y) \propto \sigma^{-n}exp\left[-\frac{1}{2\sigma^2}\left(y'\widehat{\Omega}\widehat{\Psi}y - 2\beta_R{}'X'\widehat{\Omega}\widehat{\Psi}y + \beta_R{}'X'\widehat{\Omega}\widehat{\Psi}X\beta_R - \right.\right.$$
$$\left.\left. 2\left(\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}X'\widehat{\Omega}\widehat{\Psi}y\right)'X'\widehat{\Omega}\widehat{\Psi}y + 2\left(\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}X'\widehat{\Omega}\widehat{\Psi}y\right)'X'\widehat{\Omega}\widehat{\Psi}X\left(\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}X'\widehat{\Omega}\widehat{\Psi}y\right)\right)\right]$$
$$= \sigma^{-n}exp\left[-\frac{1}{2\sigma^2}(y - Xb_R)'\widehat{\Omega}\widehat{\Psi}(y - Xb_R) + b_R{}'X'\widehat{\Omega}\widehat{\Psi}Xb_R + \beta_R{}'X'\widehat{\Omega}\widehat{\Psi}X\beta_R - 2\beta'X'\widehat{\Omega}\widehat{\Psi}Xb\right]$$
$$= \sigma^{-n}exp\left[-\frac{1}{2\sigma^2}\left(\hat{\sigma}(n-k) + (\beta_R - b_R)'X'\widehat{\Omega}\widehat{\Psi}X(\beta_R - b_R)\right)\right]$$

Setting the priors

$$p(\beta_R|\sigma^2) = (2\pi)^{-\frac{k}{2}}\left|\widehat{\Omega}\widehat{\Psi}\right|^{-\frac{1}{2}} exp\left[-\frac{1}{2}(\beta_R - \mathbb{B})'\widehat{\Omega}\widehat{\Psi}^{-1}(\beta_R - \mathbb{B})\right]$$
$$p(\sigma^2) \propto \sigma^{-(a-k)} exp\left[-\frac{b}{\sigma^2}\right]$$

Note, normal-inverse gamma priors are conjugate priors and were selected because the prior and posterior densities are of the same family of distributions Oloyede (2022).

$$p(\beta_R|\sigma^2)p(\sigma^2) = (2\pi)^{-\frac{k}{2}}\left|\widehat{\Omega}\widehat{\Psi}\right|^{-\frac{1}{2}} exp\left[-\frac{1}{2}(\beta_R - \mathbb{B})'\widehat{\Omega}\widehat{\Psi}^{-1}(\beta - \mathbb{B})\right] \times \sigma^{-(a-k)} exp\left[-\frac{b}{\sigma^2}\right] \quad (2.10)$$

$$p(\beta_R|\sigma^2)p(\sigma^2) = \sigma^{-(a-k)} exp\left[-\frac{1}{2\sigma^2}(\beta_R - \mathbb{B})'\widehat{\Omega}\widehat{\Psi}^{-1}(\beta_R - \mathbb{B}) + 2b\right] \quad (2.11)$$

$n \times k$ size of X matrix, β_R unknown parameters, $\mathbb{B}$ prior mean vector of beta (true value), σ^2 prior variance for beta, $\hat{\sigma}^2 = \frac{(y-X\beta_R)\widehat{\Omega}\widehat{\Psi}(y-X\beta_R)}{n-k}$, $a-k$ is first hyper-parameter and b second hyper-parameter.

Joint posterior density,

$$\pi(\beta_R, \sigma^2|X, y) \propto \sigma^{-n} exp\left[-\frac{1}{2\sigma^2}\left(\hat{\sigma}(n-k) + (\beta_R - b_R)'X'\widehat{\Omega}\widehat{\Psi}X(\beta_R - b_R)\right)\right] \times$$
$$\sigma^{-(a-k)} exp\left[-\frac{1}{2\sigma^2}(\beta_R - \mathbb{B})'\widehat{\Omega}\widehat{\Psi}^{-1}(\beta_R - \mathbb{B}) + 2b\right]$$

$$\propto \sigma^{-n-a+k} exp\left[-\frac{1}{2\sigma^2}\left(\hat{\sigma}(n-k) + (\beta_R - b_R)'X'\widehat{\Omega}\widehat{\Psi}X(\beta_R - b_R) + 2b + \sigma^2(\beta_R - \mathbb{B})'\widehat{\Omega}\widehat{\Psi}^{-1}(\beta_R - \mathbb{B})\right)\right]$$

In an attempt to obtain marginal distribution of β_R, there is need to have Jacobian transformation after σ^2 is replaced with s ,

$$J = \left|\frac{d}{ds}\sigma\right| = \left|\frac{d}{ds}s^{-\frac{1}{2}}\right| = \frac{1}{2}s^{-\frac{3}{2}}$$

Replace σ^2 with s

$$\propto \left(s^{-\frac{1}{2}}\right)^{-n-a+k} exp\left[-\frac{1}{2}s\left(\hat{\sigma}(n-k) + (\beta_R - b_R)'X'\widehat{\Omega}\widehat{\Psi}X(\beta_R - b_R) + 2b + s^{-1}(\beta_R - \mathbb{B})'\widehat{\Omega}\widehat{\Psi}^{-1}(\beta_R - \mathbb{B})\right)\right]\left(\frac{1}{2}s^{-\frac{3}{2}}\right)$$

Integrate w.r.t s –nuisance parameter in order to obtain β

$$= \int_0^\infty \frac{1}{2}s^{\frac{n+a-k-3}{2}} exp\left[-\frac{1}{2}s\left(\hat{\sigma}(n-k) + (\beta_R - b_R)'X'\widehat{\Omega}\widehat{\Psi}X(\beta_R - b_R) + 2b + s^{-1}(\beta_R - \mathbb{B})'\widehat{\Omega}\widehat{\Psi}^{-1}(\beta_R - \mathbb{B})\right)\right] ds$$

Recall

$$1=\int_0^\infty \frac{q^{p+1}}{\Gamma(p+1)}s^p e^{-qs}ds$$

$$\frac{\Gamma(p+1)}{q^{p+1}}=\int_0^\infty s^p e^{-qs}ds \tag{2.12}$$

where $p=\frac{n+a-k-3}{2}$

$$q=\frac{1}{2}\left(\hat{\sigma}(n-k)+(\beta_R-b_R)'X'\widehat{\Omega}\widehat{\Psi}X(\beta_R-b_R)+2b+s^{-1}(\beta_R-\mathbb{B})'\widehat{\Omega}\widehat{\Psi}^{-1}(\beta_R-\mathbb{B})\right)$$

(36) with degree of freedom as $v=n-k$

$$\pi(\beta|p,q)\propto q^{-(p+1)}=q^{-\frac{n}{2}}$$

then replace it

$$\pi(\beta|p,q)\propto\frac{1}{2}\left(\hat{\sigma}(n-k)+(\beta_R-b_R)'X'\widehat{\Omega}\widehat{\Psi}X(\beta_R-b_R)+2b+s^{-1}(\beta_R-\mathbb{B})'\widehat{\Omega}\widehat{\Psi}^{-1}(\beta_R-\mathbb{B})\right)^{-\left(\frac{n+a-k-1}{2}\right)}$$

To obtain

$$\pi(\sigma|X,y)\propto\int_0^\infty \sigma^{-n-a+k}exp\left[-\frac{1}{2\sigma^2}\left(\hat{\sigma}(n-k)+(\beta_R-b)'X'\widehat{\Omega}\widehat{\Psi}X(\beta_R-B)+2b+\sigma^2(\beta_R-\mathbb{B})'\widehat{\Omega}\widehat{\Psi}^{-1}(\beta_R-\mathbb{B})\right)\right]d\beta_R$$

$$\pi(\sigma|X,y)\propto\int_0^\infty \sigma^{-n-a+k}exp\left[-\frac{1}{2\sigma^2}\left(\hat{\sigma}(n-k)\right)\right]\times\int_0^\infty exp\left[-\frac{1}{2\sigma^2}\left((\beta_R-b_R)'X'\widehat{\Omega}\widehat{\Psi}X(\beta_R-b_R)+2b+\sigma^2(\beta_R-\mathbb{B})'\widehat{\Omega}\widehat{\Psi}^{-1}(\beta_R-\mathbb{B})\right)\right]d\beta_R$$

Through simplification we have

$$\pi(\sigma|X,y)\propto\sigma^{-n-a+k}exp\left[-\frac{1}{2\sigma^2}\left(\hat{\sigma}(n-k)\right)\right](2\pi\sigma^2)^{\frac{k}{2}}$$

$$\pi(\sigma|X,y)\propto(\sigma^2)^{-\frac{1}{2}(n+a-2k)}exp\left[-\frac{1}{2\sigma^2}\left(\hat{\sigma}(n-k)\right)\right]$$

Thus,

$$\hat{\beta}_R\sim MVN\left(\hat{\beta}_R\ ,\ \hat{\sigma}^2\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}\left[1-\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}R'\left[R\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}R'\right]^{-1}R\ \right]\ \right)$$

$$\hat{\sigma}^2\sim IG\left(\mathrm{a}_1-\frac{\mathrm{n}}{2},b_1+\frac{1}{2}\sum_{\mathrm{i=1}}^{\mathrm{n}}(\mathrm{y}_i-X\beta_R)^2+2\left[R\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}R'\right]^{-1}(R\beta_R-r)'(r-R\beta_R)\right)$$

$$\check{\beta}_{RS}\sim MVN\left(\hat{\beta}_{RS}\ ,\ \hat{\sigma}^2\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}\left[1-\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}R'\left[R\left(X'\widehat{\Omega}\widehat{\Psi}X\right)^{-1}R'\right]^{-1}R\ \right]\ \right)$$

for Restricted Stein-rule estimator

2.2 Data Generation Processes

The Monte Carlo simulation algorithm was adopted to examine the small sample properties of the family of restricted least squares estimators with intertwining non-spherical disturbances both in classical and Bayesian frameworks. Data were generated based on the following parameter of the model: $P = 6, n = 25, y_t = X_t\beta + u_t,\ t = 1, \dots, 25$, where u_t assumed to be generated by intertwine heteroscedastic error and AR(1) process where $u_t = \rho u_{t-1} + \varepsilon_t$ or heteroscedastic error and MA(1) process where $u_t = \varepsilon_t - \rho\varepsilon_{t-1}$, $\varepsilon_t \sim N(0,1)$, $\hat{\beta}$ was set as $(1.2, 2, 0.8, 0.3, 2.1, 1.1)$ while seed was set at 1234. 10000 iterations were set for both classical and Bayesian Monte Carlo simulation. ρ was set at $-0.8, -0.5, -0.3, 0,\ 0.3, 0.5, 0.8$ for both AR(1) process and MA(1) process. Heteroscedasticity was of four categories: $\delta = 0$ No heteroscedasticity, $\delta = 0.3$ mild heteroscedasticity, $\delta = 0.5$ moderately heteroscedasticity, and $\delta = 1$ severe Heteroscedasticity. The restriction of parameters were set as

$$R = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$r = (0 \quad 1 \quad 0)$ where $\beta_1 - \beta_3 = 0, \beta_2 + \beta_4 = 1$ and $\beta_5 = 0$.

The relative efficiency was computed for each estimator as $\frac{R(\hat{\beta}_R)}{R(\hat{\beta}_{RS})}$, the value of relative efficiency greater than one implies that $\hat{\beta}_{RS}$ is more efficient compared with $\hat{\beta}_R$. All computations were carried out using R (2022) Statistical software. The dataset class contained the posterior sample for the model parameters.

3. Quadratic Loss and Risk Function

Quadratic weight loss and risk function that incorporated both error structures were used to evaluate the performances of classical, Bayes estimate and posterior mean. Let $L(\hat{\beta}_R - \beta) = (\hat{\beta}_R - \beta)Q(\hat{\beta}_R - \beta)$ be quadratic or square error loss function where $\hat{\beta}_R$ is the restricted parameters, β is the prior mean, and Q is the $\sum_{i=1}^{\beta_R} \hat{\beta}_R$ weight of loss function. For the comparison of $\hat{\beta}_R$, and $\hat{\beta}_{RS}$ in both classical and Bayesian paradigms, both were compared with the weighted square error loss function and it's associated relative efficiency, to capture small sample properties.

4. Data Analysis, Results and interpretation

Table 1. Restricted Stein-rule estimators with AR(1) process and heteroscedasticity

		Autoreggressive model (1) process					
ρ	δ	RLS	RLSTein	BRLSPost	BRLSBayes	BRLSTeinPost	BRLSTeinBayes
-0.8	0	11.95386	11.94617	11.76803	11.76843	11.76712	11.76682
-0.5	0	11.40917	11.41245	11.14193	11.14237	11.14090	11.14056
-0.3	0	11.02323	11.02693	10.73144	10.73188	10.73039	10.73004
0	0	10.62719	10.62887	10.32334	10.32374	10.32235	10.32201
0.3	0	10.30029	10.30007	10.00582	10.00615	10.00498	10.00467
0.5	0	10.18246	10.18213	9.88554	9.88584	9.88481	9.88453
0.8	0	10.17754	10.18209	9.78604	9.78637	9.78531	9.78504
-0.8	0.3	10.08097	10.06767	9.75367	9.75422	9.75246	9.75205
-0.5	0.3	10.89463	10.89858	10.47550	10.47623	10.47377	10.47319

-0.3	0.3	10.62769	10.63281	10.16825	10.16898	10.16648	10.16588
0	0.3	10.38082	10.38308	9.90666	9.90733	9.90499	9.90442
0.3	0.3	10.23447	10.23385	9.77023	9.77078	9.76881	9.76828
0.5	0.3	10.10466	10.10371	9.63134	9.63184	9.63009	9.62962
0.8	0.3	10.17370	10.17961	9.55151	9.55205	9.55028	9.54982
-0.8	0.5	10.44131	10.42244	10.08665	10.08748	10.08477	10.08413
-0.5	0.5	10.36106	10.36522	9.77913	9.78012	9.77672	9.77594
-0.3	0.5	10.33580	10.34205	9.71216	9.71318	9.70960	9.70877
0	0.5	10.20251	10.20509	9.56386	9.56480	9.56142	9.56060
0.3	0.5	10.13022	10.12901	9.50297	9.50376	9.50088	9.50012
0.5	0.5	10.24925	10.24750	9.59190	9.59262	9.59006	9.58935
0.8	0.5	9.67637	9.68216	8.84302	8.84381	8.84118	8.84051
-0.8	1	9.27073	9.21880	8.43962	8.44199	8.43283	8.43101
-0.5	1	10.24926	10.25441	9.10058	9.10333	9.09230	9.09005
-0.3	1	10.96178	10.97710	9.71060	9.71348	9.70181	9.69933
0	1	10.36995	10.37623	9.02841	9.03092	9.02065	9.01834
0.3	1	9.66596	9.66111	8.41505	8.41708	8.40889	8.40689
0.5	1	9.08793	9.07847	7.87155	7.87330	7.86632	7.86450
0.8	1	9.59359	9.60327	7.77654	7.77875	7.77054	7.76861

The study observed the dominance of Bayesian restricted and restricted Stern-rule estimators over classical restricted and Stein-rule estimators under weighted quadratic loss functions. A Monte Carlo simulation study was used to study the behaviour of the Bayesian and classical restricted Stein-rule estimators with respect to the intertwining of heteroscedastic and autocorrelated AR(1) process disturbance errors in small sample properties. When ρ is increasing monotonically, the risk function is decreasing, this is due to the intertwining of ρ with δ.

Comparing classical restricted least squares (RLS) and restricted Stein-rule least squares (RSRLS) with intertwine autocorrelated and heteroscedastic disturbance errors using loss and risk functions, it was observed that restricted Stein-rule least squares outperformed restricted least squares when ρ are -0.8, 0.3 and 0.5 whereas RLS outperformed RSRLS when ρ are -0.5, -0.3,0, and 0.8, all at scales of heteroscedasticity. It was observed in the study that Bayesian Restricted Stein-rule Bayes Estimate (BRSRBE) outperformed Bayesian Restricted Stein-rule Posterior Mean (BRSRPM), Bayesian Restricted Posterior Mean (BRPM), and Bayesian Restricted Bayes Estimate (BRBE) in this order across all the ρ and scale of heteroscedasticity. The study observed the superior performances of the Bayesian paradigm over the classical family of restricted least squares. It was observed that the Bayesian family of restricted least squares both posterior mean and Bayes estimate outperformed the classical family of restricted least squares when there exists intertwining of AR(1) autocorrelated and heteroscedastic disturbances errors.

Table 2. Relative Efficiency of Restricted Stein-rule estimators with AR(1) process and heteroscedasticity

ρ	δ	RLSTein	BRLSPost	BRLSBayes	BRLSTeinPost	BRLSTeinBayes
-0.8	0	1.000644	1.015791	1.015757	1.01587	1.015896
-0.5	0	0.999713	1.023985	1.023945	1.02408	1.024111
-0.3	0	0.999664	1.02719	1.027148	1.027291	1.027324
0	0	0.999842	1.029433	1.029393	1.029532	1.029566
0.3	0	1.000021	1.02943	1.029396	1.029516	1.029548

0.5	0	1.000032	1.030036	1.030005	1.030112	1.030141
0.8	0	0.999553	1.040006	1.039971	1.040084	1.040112
-0.8	0.3	1.001321	1.033557	1.033498	1.033685	1.033728
-0.5	0.3	0.999638	1.040011	1.039938	1.040182	1.04024
-0.3	0.3	0.999518	1.045184	1.045109	1.045366	1.045427
0	0.3	0.999782	1.047863	1.047792	1.048039	1.0481
0.3	0.3	1.000061	1.047516	1.047457	1.047668	1.047725
0.5	0.3	1.000094	1.049144	1.049089	1.04928	1.049331
0.8	0.3	0.999419	1.06514	1.06508	1.065278	1.065329
-0.8	0.5	1.001811	1.035161	1.035076	1.035354	1.03542
-0.5	0.5	0.999599	1.059507	1.0594	1.059769	1.059853
-0.3	0.5	0.999396	1.064212	1.064101	1.064493	1.064584
0	0.5	0.999747	1.066777	1.066673	1.06705	1.067141
0.3	0.5	1.000119	1.066006	1.065917	1.06624	1.066325
0.5	0.5	1.000171	1.068532	1.068452	1.068737	1.068816
0.8	0.5	0.999402	1.094238	1.09414	1.094466	1.094549
-0.8	1	1.005633	1.098477	1.098169	1.099362	1.099599
-0.5	1	0.999498	1.126221	1.12588	1.127246	1.127525
-0.3	1	0.998604	1.128847	1.128512	1.12987	1.130158
0	1	0.999395	1.148591	1.148272	1.149579	1.149873
0.3	1	1.000502	1.148652	1.148374	1.149493	1.149766
0.5	1	1.001042	1.154529	1.154272	1.155296	1.155564
0.8	1	0.998992	1.233658	1.233307	1.23461	1.234917

In Table 2 above, the relative efficiency was computed for each estimator as $\frac{R(\hat{\beta}_R)}{R(\hat{\beta}_{RS})}$, the value of relative efficiency greater than one implies that $\hat{\beta}_{RS}$ is more efficient and preferred to $\hat{\beta}_R$, this occurred across board, the study observed keenly that the Bayesian family of restricted least squares estimators are more efficient and preferred to the classical family of restricted least squares estimators with autocorrelated AR(1) process heteroscedastic disturbance errors.

Table 3: Restricted Stein-rule estimators with MA(1) process and heteroscedasticity

	Moving Average						
ρ	δ	RLS	RLSTein	BRLSPost	BRLSBayes	BRLSTeinPost	BRLSTeinBayes
-0.8	0	11.59345	11.59000	11.34210	11.34271	11.34066	11.34020
-0.5	0	11.08002	11.07910	10.85084	10.85123	10.84992	10.84962
-0.3	0	10.85754	10.85759	10.62377	10.62415	10.62288	10.62258
0	0	10.62719	10.62887	10.32334	10.32374	10.32235	10.32201
0.3	0	10.43056	10.43392	9.91997	9.92046	9.91871	9.91826
0.5	0	10.38246	10.38662	9.59600	9.59662	9.59437	9.59378
0.8	0	10.13854	10.14206	9.16135	9.16247	9.15821	9.15708
-0.8	0.3	11.52695	11.52137	11.04155	11.04253	11.03927	11.03852
-0.5	0.3	11.11431	11.11282	10.73870	10.73933	10.73719	10.73669
-0.3	0.3	10.86169	10.86167	10.49059	10.49119	10.48912	10.48862
0	0.3	10.38082	10.38308	9.90666	9.90733	9.90499	9.90442
0.3	0.3	10.28248	10.28752	9.49084	9.49167	9.48861	9.48783
0.5	0.3	10.32635	10.33275	9.09756	9.09863	9.09456	9.09351

0.8	0.3	11.06234	11.06848	9.69412	9.69587	9.68889	9.68703
-0.8	0.5	10.94189	10.93463	10.03172	10.03306	10.02855	10.02751
-0.5	0.5	11.67814	11.67675	11.19570	11.19661	11.19352	11.19278
-0.3	0.5	10.60190	10.60125	10.07790	10.07875	10.07582	10.07512
0	0.5	10.20251	10.20509	9.56386	9.56480	9.56142	9.56060
0.3	0.5	10.22358	10.23013	9.15677	9.15798	9.15339	9.15225
0.5	0.5	10.44728	10.45579	8.75034	8.75188	8.74570	8.74417
0.8	0.5	10.33556	10.34405	8.19577	8.19844	8.18732	8.18444
-0.8	1	10.28271	10.26285	8.13586	8.13898	8.12748	8.12503
-0.5	1	10.27316	10.26809	8.89410	8.89610	8.88887	8.88723
-0.3	1	10.27946	10.27710	9.20662	9.20870	9.20084	9.19904
0	1	10.36995	10.37623	9.02841	9.03092	9.02065	9.01834
0.3	1	10.22061	10.23517	7.88021	7.88349	7.86854	7.86537
0.5	1	10.14051	10.15775	6.68047	6.68431	6.66515	6.66143
0.8	1	11.05179	11.06888	7.36278	7.36911	7.33860	7.33199

In Table 3 above where MA (1) process intertwined with heteroscedastic, the outcome of the study is in line with the works of Oloyede (2022) and Chaturvedi et al. (2021). Comparing classical restricted least squares (RLS) and restricted Stein-rule least squares (RSRLS) with interwine autocorrelated and heteroscedastic disturbance errors using loss and risk functions, it was observed that restricted Stein-rule least squares outperformed restricted least squares when ρ are -0.8, and -0.5 whereas RLS outperformed RSRLS when ρ are -0.3,0, 0.3,0.5 and 0.8, at 0 scales of heteroscedasticity. But in 0.3 to 1 scales heteroscedasticity considered in the study, restricted Stein-rule least squares outperformed restricted least squares when ρ are -0.8, -0.5 and -0.3 whereas RLS outperformed RSRLS when ρ are 0, 0.3,0.5 and 0.8. Bayesian paradigm followed the same pattern as AR(1) process heteroscedastic disturbance errors. The study observed the superior performances of Bayesian paradigm over classical family of restricted least squares. It was observed that Bayesian family of restricted least both posterior mean and Bayes estimate out performed classical family of restricted least squares when there exist interwine of autocorrelated MA(1) process and heteroscedastic disturbances errors.

Table 4: Relative Efficiency of Restricted Stein-rule estimators with MA(1) process and heteroscedasticity

ρ	δ	RLSTein	BRLSPost	BRLSBayes	BRLSTeinPost	BRLSTeinBayes
-0.8	0	1.000298	1.022161	1.022106	1.022291	1.022332
-0.5	0	1.000083	1.021121	1.021084	1.021208	1.021236
-0.3	0	0.999995	1.022004	1.021968	1.02209	1.022119
0	0	0.999842	1.029433	1.029393	1.029532	1.029566
0.3	0	0.999678	1.051471	1.051419	1.051604	1.051652
0.5	0	0.999599	1.081957	1.081887	1.082141	1.082207
0.8	0	0.999653	1.106664	1.106529	1.107044	1.10718
-0.8	0.3	1.000484	1.043961	1.043869	1.044177	1.044248
-0.5	0.3	1.000134	1.034977	1.034917	1.035123	1.035171
-0.3	0.3	1.000002	1.035375	1.035315	1.03552	1.035569
0	0.3	0.999782	1.047863	1.047792	1.048039	1.0481
0.3	0.3	0.99951	1.083411	1.083316	1.083666	1.083755
0.5	0.3	0.999381	1.135068	1.134935	1.135443	1.135574
0.8	0.3	0.999445	1.141139	1.140933	1.141755	1.141974

-0.8	0.5	1.000664	1.090729	1.090584	1.091074	1.091187
-0.5	0.5	1.000119	1.043092	1.043007	1.043295	1.043364
-0.3	0.5	1.000061	1.051995	1.051906	1.052212	1.052285
0	0.5	0.999747	1.066777	1.066673	1.06705	1.067141
0.3	0.5	0.99936	1.116505	1.116358	1.116917	1.117056
0.5	0.5	0.999186	1.193928	1.193718	1.194562	1.194771
0.8	0.5	0.999179	1.261085	1.260674	1.262386	1.26283
-0.8	1	1.001935	1.263875	1.26339	1.265178	1.26556
-0.5	1	1.000494	1.155053	1.154794	1.155733	1.155946
-0.3	1	1.00023	1.116529	1.116277	1.117231	1.117449
0	1	0.999395	1.148591	1.148272	1.149579	1.149873
0.3	1	0.998577	1.296997	1.296458	1.298921	1.299444
0.5	1	0.998303	1.517934	1.517062	1.521423	1.522272
0.8	1	0.998456	1.501035	1.499746	1.505981	1.507338

In Table 4, similar to table 4.2, the relative efficiency was computed for each estimator as $\frac{R(\hat{\beta}_R)}{R(\hat{\beta}_{RS})}$, the value of relative efficiency greater than one implies that $\hat{\beta}_{RS}$ is more efficient and preferred to $\hat{\beta}_R$, this occurred across board, the study observed keenly that Bayesian family of restricted least squares estimators are more efficient and preferred to the classical family of restricted least squares estimator with autocorrelated MA(1) process heteroscedastic disturbance errors.

5. Conclusion

The study introduced a novel Bayesian restricted Stein-rule with intertwined autocorrelated AR (1), MA(1), and heteroscedastic disturbance errors, and compared it with the classical family of restricted least squares. The study observed the outperformance and relative efficiency of the Bayesian family of restricted least squares estimators over the classical family of restricted least squares. The study observed the superior performances of the Bayesian paradigm over the classical family of restricted least squares. It was observed that the Bayesian family of restricted least squares both posterior mean and Bayes estimate outperformed the classical family of restricted least squares when there exists intertwining of AR(1)/MA(1) autocorrelated and heteroscedastic disturbances errors.

It is recommended that whenever analyst and researcher having data and/or model with dual disturbance errors (autocorrelated and heteroscedastic errors), they should consider Bayesian restricted framework, this is due to its probabilistic nature of Bayesian inference.

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