Malaysian Journal of Science **30** (3): 217-221(2011)

THE UPPER DOUBLE GEODETIC NUMBER OF A GRAPH

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Received on 30th May 2011, accepted in revised form 16th Dec 2011

ABSTRACT For vertices x and y in a connected graph G of order n, the distance $d(x,y)$ is the length of a shortest x-y path. An x-y path of length $d(x,y)$ is called an x-y geodesic. The closed interval $I[x,y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V, I[S] = \bigcup_{x,y\in S} I[x,y]$. A set S of vertices in G is called a double geodetic set of G if for each pair of vertices x,y there exist vertices $u,v\in S$ such that $x,y\in I[u,v]$. The double geodetic number $dg(G)$ is the minimum cardinality of a double geodetic set. Any double geodetic set of cardinality $dg(G)$ is called *dg-set* of G. A double geodetic set in a connected graph G is called a minimal double geodetic set if no proper subset of S is a double geodetic set of G. The upper double geodetic number $dg^+(G)$ of G is the maximum cardinality of a minimal double geodetic set of G. The upper double geodetic numbers of certain standard graphs are obtained. It is proved that for a connected graph G of order n, $dg(G) = n$ if and only if $dg^+(G) = n$. It is also proved that $dg(G) = n-1$ if and only if $dg^+(G) = n-1$ for a non-complete graph G of order n having a vertex of degree $n-1$. For every two positive integers a and b, where $2 \le a \le b$, there exists a connected graph G with $dg(G) = a$ and $dg^+(G) = b$.

(Keywords: double geodetic set, double geodetic number, upper double geodetic set, upper double geodetic number)

INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m, respectively. For basic graph theoretic terminology, we refer to Harary [4]. A vertex v is said to *lie on* an x-y geodesic P if v is a vertex of P including the vertices x and y. For any vertex u of G, the *eccentricity* of u is $e(u) = \max\{d(u, v) : v\in V\}$. A vertex v is an *eccentric vertex* of u if $e(u) = d(u, v)$. The *radius* *rad G* and *diameter diam G* of G are defined by $rad\ G = \min\{e(v) : v \in V\}$ and $diam\ G = \max\{e(v) : v \in V\}$, respectively. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v *is* an *extreme vertex* of G if the subgraph induced by its neighbors is complete. Weak extreme vertices are introduced in [8]. A vertex v in a connected graph G is called a *weak extreme vertex* if there exists a vertex u in G such that $u, v\in I[x,y]$ for a pair of vertices x, y in G, then $v = x$ or $v = y$. It is observed that each extreme vertex of a graph is weak extreme. For the graph G in Figure 1, it is clear that the pair v_2, v_5 lies only on the $v_2 - v_5$ geodesic and so v_2 and v_5 are weak extreme vertices of G. It is easily seen that each vertex of G is weak extreme.

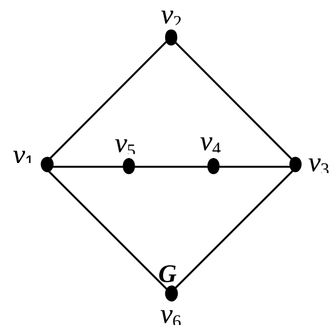

Figure 1

The *closed interval* $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x,y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set* of G. A geodetic set S in a connected graph G is a *minimal geodetic set* if no proper subset of S is a geodetic set of G. The *upper geodetic number* $g^+(G)$ of G is the maximum cardinality of a minimal geodetic set of G. The geodetic number of a graph was introduced in [1, 5] and further studied in [2, 3, 6]. It was shown in [5] that determining the geodetic number of a graph is an NP-hard problem.

A set S of vertices in G is called a *double geodetic set* of G if for each pair of vertices x, y there exist vertices u,v in S such that $x, y \in I[u,v]$. The *double geodetic number* $dg(G)$ is the minimum cardinality of a double geodetic set. Any double geodetic set of cardinality $dg(G)$ is called *dg-set* of G. A double geodetic set in a connected graph G is called a *minimal double geodetic set* if no proper subset of S is a double geodetic set of G. The *upper double geodetic number* $dg+(G)$ of G is the maximum cardinality of a minimal double geodetic set of G. The double geodetic number of graph was introduced and studied in [8]. A detailed study of double geodetic number of a graph is found in [8]. The following theorems will be used in the sequel.

Theorem 1.1. [3] Every geodetic set of a graph G contains its extreme vertices. In particular, if the set of extreme vertices S of G is a geodetic set of G, then S is the unique minimum geodetic set of G.

Theorem l.2. [3] Let G be a connected graph with a cutvertex v. Then every geodetic set of G contains at least one vertex from each component of $G - v$.

Theorem 1.3. [8] No cutvertex of a connected graph of G belongs to any minimum double geodetic set of G.

Theorem 1.4. [8] Every double geodetic set of a connected graph G contains all the weak extreme vertices of G. In particular, if the set W of all weak extreme vertices is a double geodetic set, then W is the unique dg-set of G.

Theorem 1.5. [8] For the complete bipartite graph $G = K_{m,n}(m, n \geq 2)$, $dg(G) = \min\{m,n\}$.

The Upper Double Geodetic Number of a Graph

Definition 2.1. A double geodetic set in a connected graph G is called a *minimal double geodetic set* if no proper subset of S is a double geodetic set of G. The *upper double geodetic number* $dg^+(G)$ of G is the maximum cardinality of a minimal double geodetic set of G.

Example 2.2. For the graph G in Figure 2.1 $S = \{v_2, v_4\}$ is a double geodetic set of G so that $dg(G) = 2$. The set $S' = \{v_1, v_3, v_5\}$ is a double geodetic set of G and it is clear that no proper subset of S' is a double geodetic set of G and so S' is a minimal double geodetic set of G. It is easily verified that no 4-element subset is a minimal double geodetic set and so $dg^+(G) = 3$.

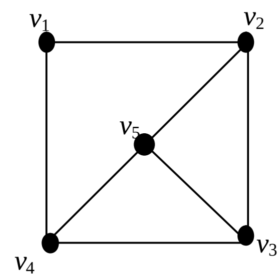

Figure 2.1

Remark 2.3. *Every minimum double geodetic set of G is a minimal double geodetic set of G and the converse need not be true. For the graph G given in Figure* 2.1, $S' = \{v_1, v_3, v_5\}$ *is a minimal double geodetic set but not a minimum double geodetic set of G.*

Theorem 2.4. *For a connected graph G of order n,* $2 \le dg(G) \le dg^+(G) \le n$.

Proof . Any double geodetic set needs at least two vertices and so $dg(G) \ge 2$. Since every minimal double geodetic set is double geodetic set, $dg(G) \le dg^+(G)$. Thus $2 \le dg(G) \le dg^+(G) \le n$.

Remark 2.5. The bounds in Theorem 2.4 are sharp. For any non-trivial path P, $dg(P) = 2$. It follows from Theorem 1.3 that $dg(T) = dg^+(T)$ for any tree T and $dg^+(K_n) = n$, $(n \ge 2)$ Also, all the inequalities in the theorem are strict. For the complete bipartite graph = $K_{r,s}$ $(3 \le r < s)$, $dg(G) = r, dg^+(G) = s$ and $n = r + s$. (See Theorems 1.5 and 2.14)

Theorem 2.6. *For a connected graph G,* $dg(G) = n$ *if and only if* $dg^+(G) = n$.

Proof. Let $dg^+(G) = n$. Then the vertex set V is the unique minimal double geodetic set of G. Since no proper subset of V is a double geodetic set, it is clear that V is also the unique minimum double geodetic set of G and so $dg(G) = n$. The converse follows from Theorem 2.4.

For the complete graph $G = K_n$, it is clear that $dg(G) = n$. Hence we have the following corollary.

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Corollary 2.7. *For the complete graph G* $= K_n$ $(n \ge 2)$, $dg^+(G) = n$.

However, a non-complete graph G of order n can have $dg(G) = dg^+(G) = n$. For the graph G given in Figure 1, all the vertices are weak extreme and so it follows from Theorem 1.4 that $dg(G) = dg^+(G) = 6$.

Theorem 2.8. *If G is a connected graph of order n with* $dg(G) = n-1$, *then* $dg^+(G) = n-1$.

Proof. Since $dg(G) = n-1$, it follows from Theorem 2.4 that $dg^+(G) = n$ or $dg^+(G) = n-1$. It follows from Theorem 2.6 that $dg^+(G) = n-1$.

A vertex in a graph G of order n is called a *full degree vertex* if its degree is $n-1$

Theorem 2.9. *Let G be a non-complete connected graph. Then a full degree vertex does not belong to any minimal double geodetic set of G*

Proof. Let S be a minimal double geodetic set of G containing a full degree vertex v_0. Let $S' = S - \{v_0\}$. We claim that S' *is* a double geodetic set of G. Let $u, v \in V$.

Case 1. $u, v \in S$. If $v_0 \ne u, v$, then $u, v \in S'$ and so S' is a double, geodetic set of G. So assume that $u = v_0$. If v is not a full degree vertex, then there exists $v' \ne v$ such that v and v' are non-adjacent and so $u, v \in I[v, v']$ with $v, v' \in S'$. Now, if v is a full degree vertex, then since the subgraph induce by S is not complete, there exist non-adjacent vertices v', v'' in S such that $u, v \in I[v', v'']$. Thus S' is a double geodetic set of G, which is a contradiction to S a minimal double geodetic set.

Case 2. $u \notin S$ or $v \notin S$. Since S is a double geodetic set, there exist $x,y \in S$ such that $u, v \in I[x,y]$. Since v_0 is a full degree vertex, it follows that $x \neq v_0$ and $y \neq v_0$. Thus $x, y \in S'$ and so S' is a double geodetic set of G, which is again a contradiction to S a minimal double geodetic set of G. Thus the proof is complete.

Theorem 2.10. *Let G be a non-complete graph of order n with a full degree vertex v. Then $dg^+(G) = n - 1$ if and only if $dg(G) = n - 1$.*

Proof. Let $dg(G) = n - 1$. Then by Theorem 2.8, $dg^+(G) = n - 1$. Let $dg^+(G) = n - 1$. Let S be a minimal double geodetic set of cardinality $n - 1$. By Theorem 2.9, $v \notin S$. Suppose that $dg(G) \leq n - 2$. Let S' be a minimum double geodetic set of G. Then it follows from Theorem 2.9 that $v \notin S'$ and $S' \subseteq S$, which is a contradiction to S a minimal double geodetic set of G. Hence $dg(G) = n - 1$.

Theorem 2.11. *Let G be a connected graph with a cutvertex v. Then every minimal double geodetic set of G contains at least one vertex from each component of $G - v$.*

Proof. This follows from Theorem 1.2.

Theorem 2.12. *No cutvertex of a connected graph G belongs to any minimal double geodetic set of G.*

Proof. Let S be any *dg-set* of G. Suppose that S contains a cutvertex z of G. Let $G_1, G_2, \ldots, G_r (r \geq 2)$ be the components of $G - z$. Let $S_1 = S - \{z\}$. We claim that S_1 is a double geodetic set of G. Let $x, y \in V(G)$. Since S is a double geodetic set, there exist $u, v \in S$ such that $x, y \in I[u,v]$. If $z \notin \{u, v\}$, then $u, v \in S_1$ and so S_1 is a double geodetic set of G, which is contradiction to the minimality of S.

Now, assume that $z \in \{u,v\}$, say $z = u$. Assume without loss of generality that v belongs to S_1. By Theorem 2.11, we can choose a vertex w in $G_k (k \neq 1)$ such that $w \in S$. Now, since z is a cut vertex of G, it follows that $I[z, v] \subseteq I[w, v]$. Hence $x, y \in I[w, v]$ with $w, v \in I[w, v]$ where $w, v \in S_1$. Thus S_1 is a double geodetic set of G, which is contradiction to the minimality of S. Thus no cut vertex belongs to any minimal double geodetic set of G.

Theorem 2.13. *For any tree T with k end-vertices $dg(T) = dg^+(T) = k$.*

Proof. This follows from Theorems 1.4 and 2.12.

Theorem 2.14. *For the complete bipartite graph $G = K_{m,n}$,*

(i) $dg^+(G) = 2$ *if* $m = n = 1$.
(ii) $dg^+(G) = n$ if $m = 1, n \geq 2$.
(iii) $dg^+(G) = \max\{m,n\}$ *if* $m, n \geq 2$.

Proof. (i) and (ii) follow from Theorem 2.13. (iii) Let X and Y be the partite sets of $K_{m,n}$. Let S be a double geodetic set of $K_{m,n}$. We claim that $X \subseteq S$ or $Y \subseteq S$. Otherwise, there exist vertices x, y such that $x \in X$, $y \in Y$ and $x, y \notin S$. Now, since the pair of vertices x, y lie only on the intervals $I[x, y]$, $I[x, t]$ and $I[s, y]$ for some $t \in X$ *and* $s \in Y$, it follows that $x \in S$ or $y \in S$, which is a contradiction to $x, y \notin S$. Thus $X \subseteq S$ or $Y \subseteq S$. It is clear that both X and Y are double geodetic sets of $K_{m,n}$ and so the result follows.

Theorem 2.15. *For any positive integers $2 \leq a \leq b$, there exists a connected graph G such that $dg(G) = a$ and $dg^+(G) = b$.*

Proof. If $a = b$, let $G = K_{1,a}$. By Theorem 2.13, $dg(G) = dg^+(G) = a$. If $a < b$, let $G = K_{a,b}$. It follows from Theorems 1.5 and 2.14 that $dg(G) = a$ and $dg^+(G) = b$.

REFERENCES

1. F. Buckley and F. Harary, *Distance in graphs*, Addison-Wesley, Redwood City, CA, 1990.

2. G. Chartrand, F. Harary and P. Zhang, *On the geodetic number of a graph*, Networks, 39(1)(2002), 1-6.

3. G. Chartrand, F. Harary, H.C. Swart and P. Zhang, *Geodomination in graphs*, Bulletin of the ICA, 31(2001), 51-59.

4. F. Harary, *Graph Theory*, Addision-Wesely, 1969.

5. F. Harary, E. Loukakis and C. Tsouros, *The geodetic number of a graph*, Mathl. Comput. Modeling, 17(11)(1993), 89-95.

6. R. Muntean and P. Zhang, *On geodomindiion in graphs*, Congr. Numer., 143(2000), 161-174.

7. P.A. Ostrand, *Graphs with specified radius and diameter*, Discrete Math., 4(1973), 71-75.

8. A.P. Santhakumaran and T. Jebaraj, *Double geodetic number of a graph*, Discuss. Math. Graph Theory, to appear.