Visualizing Algebra: Exploring a Gifted 3rd Grader's Problem-Solving Techniques for Enhanced Conceptual Understanding in Algebra Education

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Abstract
This study delves into the world of mathematical giftedness by examining the problem-solving strategies of a gifted 3rd grader in the context of algebraic equations. The research highlights the student's proficiency in creating visual representations of mathematical problems, emphasizing the potential of visualization as a valuable teaching strategy. Notably, the use of diagrams and sketches as substitutes for traditional variables like x and y has been instrumental in aiding comprehension. The gifted student's unique fluency in solving equations is noteworthy, showcasing a strong grasp of underlying principles. The paper underscores the potential for a transformative shift in algebra education by introducing visual representations at an early stage, prioritizing conceptual understanding. However, it's important to acknowledge that the study's focus on a single gifted 3rd grader might limit the broader applicability of its findings to more diverse student groups. Consideration of external elements, including the student's unique background and resource constraints, could potentially influence the observed outcomes and their generalizability. While the findings suggest that the use of visual representations may offer an effective strategy for improving algebra instruction, it is important to approach these implications with caution, recognizing the need for further research and evaluation to fully understand the impact of this approach.

Keywords: Algebra, Problem Solving, Gifted, Visualization.

Introduction

Educational research frequently employs case studies to delve deeply into individual experiences and problem-solving techniques. Drawing inspiration from Sriraman *et al.* (2013), this investigation seeks to explore the problem-solving behaviours of gifted students, offering insights into their distinctive approaches. These students invest significant time in comprehending mathematical problems before proceeding to solutions, demonstrating a proficiency in utilizing specific instances to develop generalizations and aptly employing analogies to elucidate commonalities across diverse problems. The study suggests the integration of these problem-solving techniques into educational practices.

It is well-documented that middle and high school students often grapple with understanding and executing the necessary procedures for solving simultaneous equations (Ng, 2022), which can impede their accuracy in setting up and solving such equations. Challenges also emerge in the translation of real-life scenarios into algebraic equations, frequently leading to formulation errors, and in the discernment of pertinent information from problem descriptions (Jonsson *et al*., 2020).

In addition, research in Brunei (Islami, 2022) indicates a prevalent focus on instrumental understanding among teachers, often neglecting relational understanding in favour of procedural methods, a trend observed globally in mathematics textbooks. This direct instructional approach frequently leads to misinterpretation among struggling learners due to a lack of comprehension of underlying thought processes. Ugboduma (2006, 2012) stresses the importance of effective methodologies in enhancing participant understanding of simultaneous equations. Understanding and interpreting algebraic symbols, structures, equivalent equations, and word problems, compounded by algebra's abstract nature, pose challenges for students (Kieran, 2013; Koedinger & Nathan, 2004). Similarly, studies in Zimbabwe and various regions, including Indonesia, Ontario, the Midwestern United States, and Malaysia, have uncovered widespread struggles with algebraic material, stemming from difficulties in grasping foundational algebraic concepts and honing problem-solving skills (Chirove & Ogbonnaya, 2021; Kobandaha *et al*., 2019; Noto *et al.,* 2020). While a fun learning approach has been suggested for addressing mathematical challenges and other subjects (Azman *et. al.,* 2021; Hassan *et. al*., 2021a; Hassan *et. al*., 2021b), a deeper comprehension of mathematical problems among school students may necessitate a creative approach.

These studies underscore the need for a comprehensive strategy that emphasizes conceptual understanding and innovative approaches, grounded in the experiences of mathematically gifted students. For an instance, Polya's problem-solving model (1962), still widely utilized today, serves as a framework for many research studies on problem solving. This model defines mathematical proficiency as a process that promotes independent thinking, originality, and creativity when approaching problem-solving tasks. On the other hand, Lester's research efforts have identified limitations in the metacognitive approach. As a result, Lester adapted Polya's model by incorporating cognitive elements and emphasizing the importance of documenting thought processes during problem-solving (Sriraman *et al.,* 2013). The study recognizes that both procedural and conceptual errors are prevalent among secondary school learners when learning algebraic simultaneous equations (Yunus *et al.,* 2016).

Past studies shed light on effective teaching methods and tailored programs for gifted students in algebraic problem-solving (Yang *et al.,* 2023; Deringol & Davasligil, 2020; VanTassel-Baska *et al.,* 2021). This research emphasized the importance of high expectations, critical thinking activities, active engagement, and language support for math success among gifted learners (Yang *et al.,* 2023). Similarly, another study highlighted the need for personalized teaching to improve gifted students' attitudes toward math (Deringol & Davasligil, 2020). Additionally, findings demonstrated how differentiated teaching helps gifted students in math, urging more focused use of these strategies in classrooms (VanTassel-Baska *et al.,* 2021). Despite these insights, there's room to explore specific strategies directly linked to gifted

students' problem-solving skills in algebra, aiming for a more targeted discussion on effective teaching methods for this group.

The objectives of this study are twofold: to delve into the problem-solving techniques of mathematically gifted students and, more significantly, to explore the potential implications of their approaches for a broader spectrum of learners. This article proposes an alternative teaching method inspired by a year 3 gifted student, with the belief that it has strong potential to minimize misconceptions and errors through analytical thinking.

Methodology

To ensure the research is academically rigorous, the study adheres to Yin's (2009) case study framework, starting with the definition of the primary unit of analysis - the 'case'. In this context, the research concentrates on the creative application of mathematical concepts and strategies by gifted students. Following Yin's recommendation, subunits of cases are nested within the primary case, which includes specific simultaneous equation problems successfully solved by gifted students through their unique strategies. This approach authenticates the investigation of these students' problem-solving methods, particularly focusing on a gifted student named Walter (pseudo name). This case study methodology provides insights into the cognitive processes and mathematical giftedness of this individual. Walter had been under observation and mentorship for eight months before interviews and data collection took place.

Background of Participant

Walter, a 9-year-old student attending a prestigious private school in Melbourne, has already made substantial academic strides, showcasing exceptional intellect and an unwavering commitment to learning. Despite his third-grade status, he has earned accolades for his academic achievements, including representing Melbourne in the Mathematics Olympiad at regional and state levels and gaining membership in the renowned MENSA society. Walter's accomplishments emphasize the importance of recognizing and nurturing gifted students at an early age and understanding their unique approaches to comprehending the abstract realm of mathematics.

Data Collection

This study employed a qualitative approach to delve into the 'how' of problem-solving, meticulously crafting algebraic tasks from Terry Chew's esteemed Mathematical Olympiad series, a recognized resource authored by a prominent figure and coach in Singapore. Careful selection of these tasks aimed to unveil developmental trajectories in the scrutinized skills and address our core research inquiry. The open-ended nature of these queries facilitated diverse connections across mathematical topics. Over three weeks, structured one-hour problem-solving sessions were conducted, designed to transcend mere computation, emphasizing fundamental mathematical concepts. Walter's solutions and problem-solving beliefs were systematically recorded in a folio book, capturing his approach.

Moreover, we conducted informal, unstructured interviews—an approach elucidated by Yin (2011) with the primary participant. These discussions were aimed at unravelling Walter's problem-solving techniques and cognitive patterns. Through

these interactions, we gained nuanced insights into his problem-solving strategies, enriching our understanding of his approach.

Curriculum Framework

The writer devised a mathematics curriculum that prioritizes conceptual exploration over rote answers. Walter's outstanding performance in the class, outshining his peers in mathematical aptitude, prompted this shift in teaching philosophy. While there may be concerns regarding the generalizability of analyzing the strategies of a single student, this study offers valuable insights into the cognitive processes of gifted students. Consequently, an in-depth examination of a gifted student's problem-solving strategies can enhance the understanding of mathematical abilities among other students in the classroom. If the innovative problem-solving strategies employed by this gifted student prove effective, they hold the potential to benefit a larger group of learners when implemented appropriately.

Theoretical Framework

The study aligns with Vygotsky's notion of creativity, which underscores the role of the environment in nurturing creative potential. The responsibility of fostering Walter's curiosity and engaging him in diverse problem-solving activities is acknowledged in the role of a tutor and researcher. Koshy *et al.* (2009) underscore the importance of providing challenging tasks to mathematically gifted students, in line with Vygotsky's framework. Within this study, the focus is on creativity and problem-solving in mathematics education, particularly in the context of simultaneous equations. The study places a premium on assessing the validity and reliability of research methods used in case studies.

Data Analysis

The study evaluates the creativity in problem-solving through fluency, flexibility, and originality, as proposed by Leikin & Lev (2013). Fluency assesses Walter's ability to generate numerous ideas and establish connections and associations using basic mathematical skills. Flexibility is the capacity to adapt and employ various ideas interchangeably in problem-solving. Originality pertains to Walter's ability to devise fresh and unique solutions.

In summary, the research adheres to Yin's case study framework, with a focus on Walter's exceptional problem-solving abilities. The study is grounded in Vygotsky's concept of creativity, and data analysis incorporates fluency, flexibility, and originality to assess problem-solving strategies.

Results and Discussion

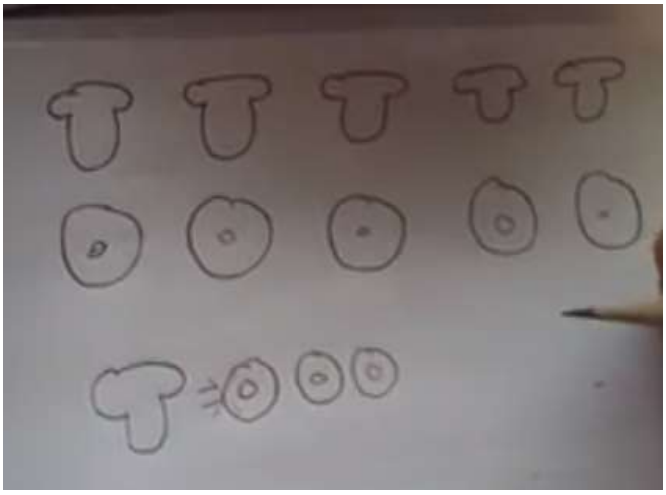
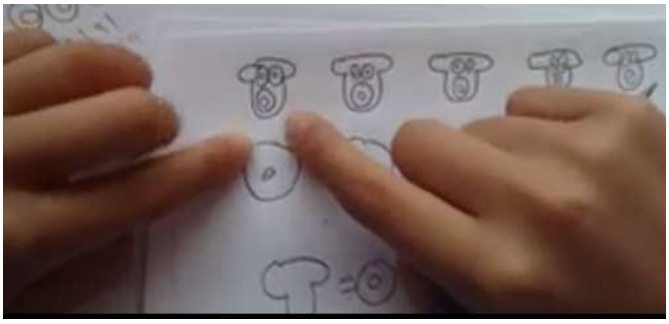
The main findings were organized into a table to that included mathematical problems and solutions provided by Walter to ensure a smooth analysis. The tutor's role in questioning and revealing the cognitive process during problem-solving sessions was also acknowledged. This detailed data analysis offers valuable insights into Walter's distinctive problem-solving approach and allows for a clearer interpretation of the results. Visual representations of Walter's strategies were deliberately included to enhance understanding of his creative problem-solving techniques.

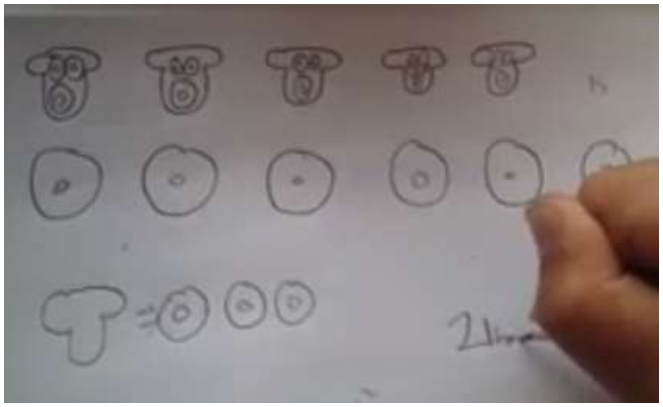
It is worth noting that when answering the simultaneous equations questions, Walter showed a high level of fluency but lacked flexibility. This means that he was able to solve all the tasks accurately without any errors, which demonstrates his strong conceptual thinking skills.

For an instance, please note his fluency in the following Problem 1:

5 raisin rolls and 6 donuts cost $14.70. Each raisin roll cost 3 times as much as each donut. How much is a raisin roll? How much is a donut?

Walter's method in solving this Problem 1 is illustrated in Figure 1.His aptitude for solving algebraic simultaneous equations without dependence on conventional techniques is evident in his expertise. He does not possess the capacity to solve simultaneous algebraic equations using the customary representation with variables x and y. Walter employed a self-devised visual method that enables him an immediate duplicate of a mathematical statement as presented in the problem-solving inquiry. Previous study highlighted that mathematically gifted students demonstrate superior analytical skills when solving problems, indicating their deep understanding of the task (Taliha & Yeliz ,2022).

	Walter directly drew the exact number and pictures of raisin rolls and doughnuts as indicated in the questions to represent the first equation He then formed the second equation by understanding using a double headed arrow to show every 1 raisin roll is equivalent to three doughnuts
	He then drew 3 doughnuts on every raisin roll and mentioned that now 'easy to divide when all are just same doughnuts'



Immediately he saw 21 doughnuts to be divided equally with the amount of money spent

Figure 1: Student's Answer Sheet for Problem 1

However, Walter's approach to flexibility leaned heavily on seeking validation from the researcher, who also played the role of his tutor. This reliance potentially restricted the exploration of alternative problem-solving methods, establishing a routine where his focus shifted once answers were confirmed. His strong confidence in his choices occasionally hindered the exploration of different strategies. Despite this tendency, Walter demonstrated flexibility by employing diverse solution strategies across other problem sets, suggesting an ability to adapt.

Nevertheless, the study's focus on a single participant limits broader generalizability, accentuating the importance of cautious interpretation. Yet, despite limitations in flexibility, Walter's confidence and motivation shone through his detailed explanations during the problem-solving process, signifying a deep comprehension of the subject matter.

Further exploration into factors contributing to Walter's limited flexibility could involve independent assessments by researchers without prior participant interaction. A suggested approach includes conducting pilot studies to evaluate problem-solving patterns before involving researchers, mitigating potential internal biases and ensuring a more impartial assessment of flexibility within the study's context.

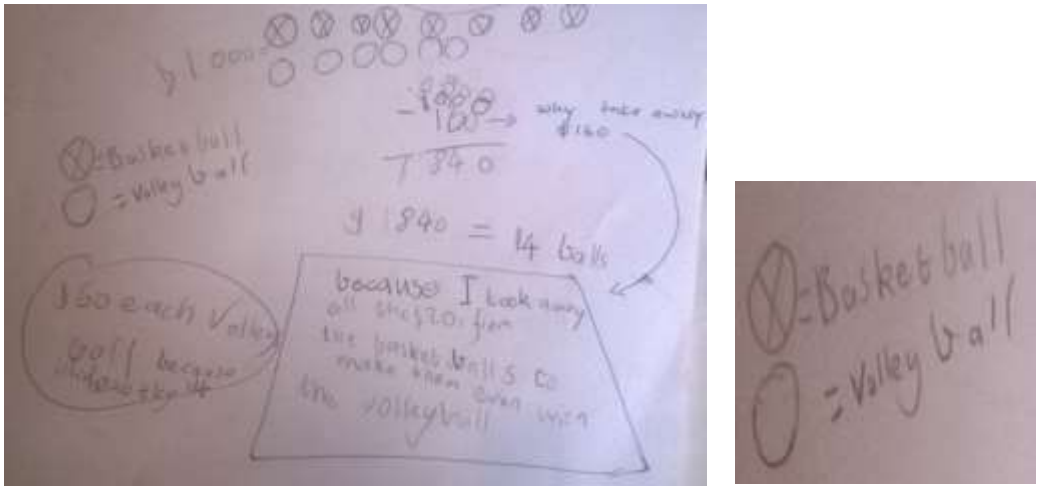
Following prior research by Matsko & Thomas (2014), Pehkonen (1997), this study supported that creativity in problem solving can be demonstrated through conceptual understanding rather than relying solely on sequential rules and algorithms, which is referred to as procedural understanding. In this study, a closer examination of the concept of originality allowed the writer to observe Walter's problem-solving of simultaneous equations process as dynamic, non-linear, and adaptable. This approach is not commonly seen among students his age or emphasized in the curriculum or textbooks.

Visual representation of the algebraic equations can surely aid in understanding and solving complex problems. Pictorial approaches in solving algebraic equations offer a more visual illustrative representation of the unknowns compared to the traditional x and y algebraic representation. Walter's approach allows for a more intuitive understanding of unknowns. His originality is displayed by using sensible pictures that were able to give meaning to unknowns in algebraic equations.

Another interesting finding that differs from what he regularly did was captured in the solution below. The question for Problem 2 as follows:

> 8 basketballs and 6 volleyballs cost $ 1000.
> A basketball costs $20 more than volleyball.
> What is the cost of volleyball?

While Walter's use of pictures was effective, he encountered a limitation when trying to represent two different items such as a volleyball and basketball with identical round circles. He realized that his visual representations did not accurately depict the distinct characteristics of each object. This highlighted the inability of his approach to capture certain aspects of algebraic problems. To address this issue, Walter proposed using crossed marks on one circle to differentiate it from the other and indicated what each symbol represents (see below).



(a) (b)

Figure 2: (a) Student's Answer Sheet for Problem 2; (b) Proposed Symbol by The Students (Enlarged)

Walter's approach of using picture representations to solve algebraic equations and give meaning to unknowns is a creative and innovative method. In summary, Walter's approach offered a creative approach that showcased his fluency and originality. Walter's diagrammatic approach to solving algebraic equations offers a unique perspective by providing visual representations and giving meaning to unknowns through sensical pictures. By using diagrams, Walter can visualize and interpret mathematical equations more tangibly. This not only helps him grasp the problem more effectively but also facilitates his problem-solving process. Furthermore, this approach promotes a deeper understanding of the underlying concepts of unknowns and relationships within the equations. It is obvious evaluation, that the notion of fluency and flexibility is parallel to what is proposed by Leikin & Lev (2013) in their model but originality to a problem solving is extended to a solution which is non-conventional.

Therefore, taking a bold step away from the norm, this model introduces sketched images as substitutes for conventional variables like x or y. Instead of relying on abstract symbols, the model utilizes easily interpretable drawings to directly

represent and solve problems. This diagrammatic approach not only enhances comprehension of algebraic problems but also establishes a visual link between variables and their connections, making it simpler for Walter to derive meaningful solutions. This study further supports the notion that gifted students employ a wider range of strategies compared to their non-gifted peers (Ollinger *et al.*, 2006). These les s utilized strategies reflect the more developed knowledge base of gifted students. The ability to generate such strategies provides stronger evidence for the effectiveness of heuristic methods in problem solving. Even without expertise in unknowns, algebraic expressions, or equations, he is still able to employ an intuitive approach that produces logical outcomes.

Walter utilizes visual representations to directly translate word problems into picture solutions, bypassing the interference brought about by traditional algebraic symbols such as x and y. The diagrammatic approach serves as an effective introduction strategy for solving problems that extends beyond traditional plug-and-chug the x and y methods. Therefore, unknowns should be earlier introduced as visual representation from the given problem before it becomes a symbol that actually make sense to learners.

Conclusion

The findings of this study align with the objectives of examining a gifted student's problem-solving techniques when tackling simultaneous equations. Walter, a third-grade student, demonstrated an exceptional level of fluency in his problem-solving approach. His ability to resolve mathematical problems accurately highlighted strong conceptual thinking skills. Furthermore, Walter's unique strategy, involving the visual representation of mathematical concepts instead of the conventional variables x and y, showcases originality and creativity in problem solving. This creative approach enhances the understanding of unknowns and the relationships within algebraic equations, making it a promising alternative to traditional methods. By introducing visual representations early in mathematical education, students can develop a deeper comprehension of algebraic concepts.

Walter's methodology, although not without limitations, offers a fresh perspective on teaching algebraic simultaneous equations. It addresses the challenge of effectively communicating problem-solving strategies to students, transforming the way these concepts are introduced. The use of visual representations enables students to grasp and interpret mathematical equations more tangibly. While it is evident that fluency and flexibility in problem solving are consistent with established models, Walter's originality introduces non-conventional solutions.

In conclusion, Walter's innovative approach, using visual representations to replace the traditional x and y, offers a promising pathway to enhance students' understanding of algebraic equations, especially in the context of simultaneous equations. This method encourages students to adopt a more intuitive and creative approach, emphasizing comprehension over rote procedures. It has the potential to provide meaningful results and improve overall comprehension, making algebraic education more accessible and engaging.

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